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Computing the split domination number of grid graphs

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Abstract

A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set D. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set* if the induced graph $\langle V - D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set*. In this paper we have introduced a new method to obtain the split domination number of grid graphs by partitioning the vertex set in terms of star graphs and also we have obtained the exact values of $\gamma_s(G_{m,n}), m \leq n, m, n \leq 24$.

Keywords: separate, domination number, split domination, grid graph, star graph
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1. Introduction

The graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual n and q denote the number of vertices and edges of a graph G. For any undefined term or notation in this paper can be found in Harary [2].

A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set D [5]. V.R.

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Kulli and B. Janakiram had introduced a concept of split domination [3]. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set*, if the induced graph $\langle V - D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set*.

A two dimensional grid graph $G_{m,n}$ is the graph Cartesian product $P_m \times P_n$ of paths on m and n vertices. The Cartesian graph product of $G_1 \times G_2$ with disjoint vertex sets and edge sets in G_1, G_2 is the graph with the vertex set $V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and $u_2$adj $v_2]$ or $[u_2 = v_2$ and $u_1$adj $v_1]$. A star graph is a complete bipartite graph of the form $K_{1,n-1}$ with n vertices. The neighborhood of a vertex in the graph G is the set of vertices adjacent to v and is denoted by $N(v)$.

Computing of domination of grid graph has been studied in $[1, 4]$. In this paper we have introduced a new method to obtain the split domination number of a grid graphs by partitioning the vertex set in terms of $K_{1,3}, K_{1,2}, K_2$ and K_1 and also we have obtained the exact values of $\gamma_s(G_{m,n}), m \leq n, m, n \leq 24$.

2. Preliminaries

To simplify the description of the algorithm, we first define an order of the vertices of an mn grid graph with vertices $v_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq n$. Every minimum dominating set can be constructed by an exhaustive search where in each step any undominated vertex is picked, after which all possible ways of dominating this vertex are considered in turn.

The procedure to construct the minimum split dominating set is as follows:
Pick the vertex of degree 3 say v_i, $E = v_i \cup N(v_i)$. Next choose the vertex in $V(G) - E$, that has the vertex of degree 3, if not choose the vertex of degree 2 otherwise choose the vertex of degree 1 say v_j, if not choose $v_k \in E$. Let $D = v_i \cup v_j \cup v_k$ such that the number of vertices in D is minimum and $< V(G) - D >$ is disconnected, this procedure is continued unless $V(G) - E$ is an empty set. Suppose if $< V(G) - D >$ is connected, then we need one more vertex to make the graph disconnected.

3. Algorithm to Find the Split Domination Number of grid graph by partitioning the vertex set.

Step 1: Divide the grid graph $G_{m,n}$ by partitioning the vertex set in terms $K_{1,3}, K_{1,2}, K_{1,1}$ and K_1 such that

(i) Partitioning $A = \{P_1, P_2, P_3......P_n\}$ is minimum.
(ii) $< V(P_1) > \cap < V(P_2) > \cap ....\cap < V(P_n) >= \phi$ and $v_1 \in < V(P_1) >$, $v_2 \in < V(P_2) >$ if $deg(v_1)$ is maximum in $< V(P_1) >$ and $v_2 \notin N(v_1)$ in $G_{m,n}$.
(iii) $E(P_1) \cap E(P_2) = \phi, E(P_1)$ is the edges in $< V(P_1) >$.

Step 2: Suppose A contains atleast two partition set say P_1, P_2 such that $< V(P_1) >= K_1$ and $< V(P_2) >= K_1$ and $H = \{v_c \in N(V(P_1) \cap V(P_2))$ in $G_{m,n}\}$, $A = \{A - \{P_1, P_2\}\}$.

Step 3: Let $D = \{v_i \in < V(P_i) >, P_i \in A/v_i$ is of max deg of $< V(P_i) >$ and $< V(P_i) > \neq K_{1,1}\}$

Step 4: IF A contains atleast one partition P_j, such that $< V(P_j) >= K_{1,1}$ with $< V(P_j) >= \{v_m, v_r\}$.
GOTO STEP 5.
ELSE IF $H = \phi$
$C = D$
ELSE
$C = D \cup H$
GOTO STEP 7.

Step 5: For each partition P_j.
IF atleast one vertex say $v_s, v_s \in N(v_m), v_s \notin < V(P_j) >$ and $N(v_s) - \{v_m\} \in D$.
$\{v_m\} \in S$.
ELSE
$\{v_m\}$ or $\{v_r\} \in S$.
GOTO STEP 5.
Let S be the set of all such vertices.

Step 6: IF $H = \phi$ and $S = \phi$
$C = D$
ELSE IF $H = \phi$ and $S \neq \phi$
$C = D \cup S$.
ELSE
$C = D \cup S \cup H$.

Step 7: Find the split adjacency matrix

$$a_{ij} = \begin{cases} v_{ij} = 1 & v_i \text{ is adjacent to } v_j \text{ and } v_i \notin C \\ v_{ij} = 0 & \text{Otherwise.} \end{cases}$$

Step 8: IF a_{ij} contains atleast one zero row then, $\gamma_s = |C|$
GOTO STEP 11.
ELSE
GOTO STEP 9.

Step 9: Let $\{v_k\}$ is the row in a_{ij} in which sum of all the elements in $\{v_k\} = 1$ and 1 is present in v_p column.

Step 10: $\gamma_s = |C| + |\{v_p\}| = |C| + 1$.

Step 11: END

4. Examples

$$A = \{P_1, P_2, P_3, P_4, P_6\}$$
$$D = \{v_2, v_{11}, v_9, v_{18}, v_5\}$$

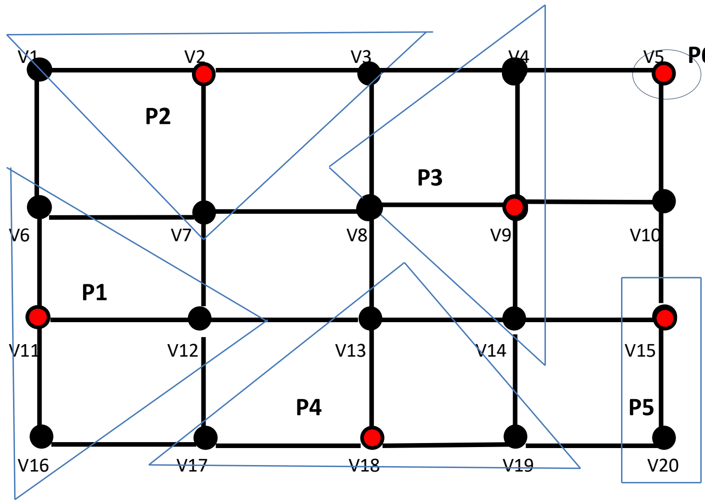

Figure 1. A grid graph $G_{4,5}$

For partition P_5,$v_s = v_{10}, v_m = 15, S = \{v_{15}\}$

$C = D \cup S = \{v_2, v_{11}, v_9, v_{18}, v_5, v_{15}\}$

The Split adjacent matrix a_{ij} is:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}
v_1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_2	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
v_3	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
v_4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_5	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
v_6	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
v_7	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0
v_8	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
v_9	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0
v_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_{11}	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0
v_{12}	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0
v_{13}	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0
v_{14}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
v_{15}	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
v_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
v_{17}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
v_{18}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
v_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
v_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

Since there exists a zero row v_{10} in a_{ij}. therefore the split domination number $\gamma_s = |C| = 6$.

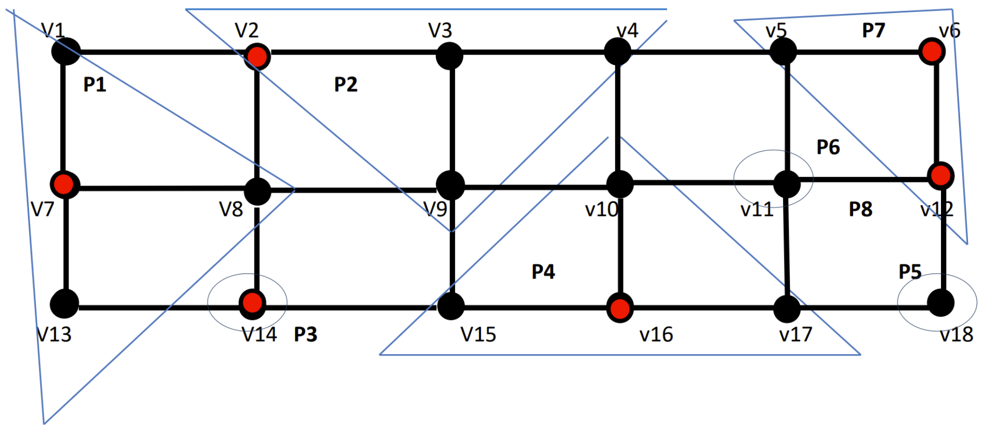

Figure 2. A grid graph $G_{3,6}$

$$A = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$$

Since $< V(P_5) >$ and $< V(P_6) >= K_1$ and $H = \{v_c = v_{12}\}$

$$A = \{P_1, P_2, P_3, P_4, P_7\}$$

$$D = \{v_3, v_6, v_7, v_{14}, v_{16}\}$$

$$C = D \cup H = \{v_3, v_6, v_7, v_{14}, v_{16}, v_{12}\}$$

The Split adjacent matrix a_{ij} is:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
v_1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_2	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
v_3	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
v_4	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
v_5	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
v_6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
v_7	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
v_8	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
v_9	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0
v_{10}	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0
v_{11}	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	0
v_{12}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
v_{13}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_{14}	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0
v_{15}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
v_{16}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0
v_{17}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
v_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

Since there exists a zero rowv_{13} in $a_{ij}, \gamma_s = |C| = 6$

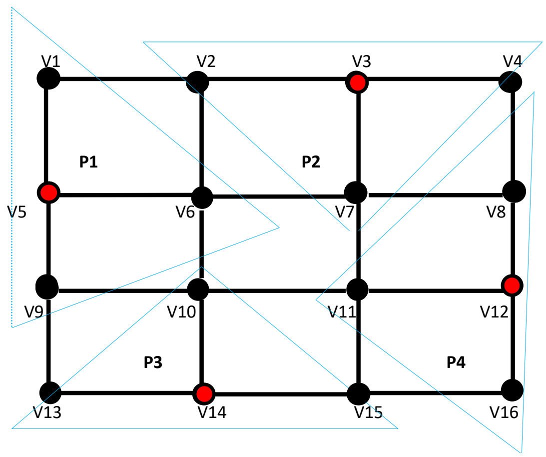

Figure 3. A grid graph $G_{4,4}$

$$A = \{P_1, P_2, P_3, P_4\}$$
$$D = \{v_3, v_5, v_{12}, v_{14}\}$$
$$H = \phi$$
$$C = D = \{v_3, v_5, v_{12}, v_{14}\}$$

The Split adjacent matrix a_{ij} is:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_2	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
v_3	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
v_4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
v_5	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
v_6	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0
v_7	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0
v_8	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
v_9	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
v_{10}	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0
v_{11}	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0
v_{12}	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1
v_{13}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
v_{14}	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0
v_{15}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
v_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

Since there exists a non-zero row in a_{ij} and v_{16} contains 1 in v_{15}^{th}column=$v_p, \gamma_s = |C| + 1 = 5$

Theorem 4.1. *For any grid graph $G_{m,n}$,*

$$\gamma_s(G_{m,n}) = \begin{cases} \gamma(G_{m,n}) & \text{if } \langle V(G_{m,n}) - \gamma(G_{m,n})\rangle \text{ is disconnected} \\ \gamma(G_{m,n}) + 1 & \text{otherwise.} \end{cases}$$

Proof. Let D be the γ-set of G.

Case 1: if $\langle V(G) - D\rangle$ is disconnected, the result follows from the definition of split dominating set.

Case 2: if $\langle V(G) - D\rangle$ is connected, then there exists atleast one vertex say v_i in D which is of degree 3 and $v_i \in N(v_j)$, v_j is of degree 2. Then $D \cup v_k$, $v_k \in N(v_j) \neq v_i$. Since v_j is of degree 2, $D \cup v_k$ is disconnected. Hence $\gamma_s(G_{m,n}) = \gamma(G_{m,n}) + 1$.

□

5. Exact values of $\gamma_s(G_{m,n}), m \leq n$.

The exact values of $\gamma_s(G_{m,n})$ are obtained by using Theorem 4.1 and [1, 4]

$$\gamma_s(G_{1,n}) = \lfloor \tfrac{n+2}{2} \rfloor$$

$$\gamma_s(G_{2,n}) = \begin{cases} \lfloor \frac{n+2}{2} \rfloor & n \text{ is even or } n = 3 \\ \lfloor \frac{n+2}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{3,n}) = \begin{cases} \lfloor \frac{3n+4}{4} \rfloor & n \equiv 0(mod4) or n = 3 \\ \lfloor \frac{3n+4}{4} \rfloor + 1 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{4,n}) = n + 1.$$

$$\gamma_s(G_{5,n}) = \begin{cases} \frac{6n+13}{5} & n = 5p + 2, p \geq 2 \\ \lceil \frac{6n+8}{5} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{6,n}) = \begin{cases} \lceil \frac{10n+10}{7} \rceil & n \equiv 1(mod7) \\ \frac{10n+19}{7} & n = 7p + 3, p \geq 1 \\ \lceil \frac{10n+12}{7} \rceil & \text{otherwise} \end{cases}$$

$$\gamma_s(G_{7,n}) = \begin{cases} \frac{5n+6}{3} & n = 3p, p \geq 3 \\ \lceil \frac{5n+3}{3} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{8,n}) = \begin{cases} \frac{15n+22}{8} & n = 8p + 6, p \geq 1 \\ \lceil \frac{15n+14}{8} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{9,n}) = \begin{cases} \frac{23n+31}{11} & n = 11p + 2, p \geq 1 \\ \lceil \frac{23n+20}{11} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{10,n}) = \begin{cases} \lceil \frac{30n+37}{13} \rceil & n \neq 13, 16, n \equiv 0, 3(mod13), n = 13p + 7, p \geq 1 \\ \lceil \frac{30n+24}{13} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{11,n}) = \begin{cases} \lceil \frac{38n+21}{15} \rceil & n = 11, 20, 22 \\ \frac{38n+51}{15} & n = 15p+3, p \geq 3 \\ \lceil \frac{38n+36}{15} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{12,n}) = \begin{cases} \frac{80n+95}{29} & n = 29p-3, p \geq 1 \\ \lceil \frac{80n+66}{29} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{13,n}) = \begin{cases} \lceil \frac{98n+111}{33} \rceil & n \equiv 14, 15, 17, 20 (mod 33), n = 33p+12, p \geq 1 \\ \lceil \frac{98n+78}{33} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{14,n}) = \begin{cases} \lceil \frac{35n+40}{11} \rceil & n \equiv 18 (mod 22), n = 11p+13, p \geq 1 \\ \lceil \frac{35n+29}{11} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{15,n}) = \begin{cases} \lceil \frac{44n+27}{13} \rceil + 1 & n \equiv 5 (mod 26), \\ \frac{44n+53}{13} & n = 13p+5, p \geq 1 \\ \lceil \frac{44n+40}{13} \rceil & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{16,n}) = \begin{cases} \frac{18n+21}{5} & n = 5p+13, p \geq 1 \\ \lceil \frac{18n+36}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{17,n}) = \begin{cases} \frac{19n+23}{5} & n = 5p+13, p \geq 1 \\ \lceil \frac{19n+38}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{18,n}) = 4n + 5.$$

$$\gamma_s(G_{19,n}) = \begin{cases} \frac{21n+27}{5} & n = 5p+13, p \geq 2 \\ \lceil \frac{21n+42}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{20,n}) = \begin{cases} \frac{22n+29}{5} & n = 5p+13, p \geq 2 \\ \lceil \frac{22n+44}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{21,n}) = \begin{cases} \frac{23n+31}{5} & n = 5p+13, p \geq 2 \\ \lceil \frac{23n+46}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{22,n}) = \begin{cases} \frac{24n+33}{5} & n = 5p+13, p \geq 2 \\ \lceil \frac{24n+48}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

$$\gamma_s(G_{23,n}) = 5n + 7.$$

$$\gamma_s(G_{24,n}) = \begin{cases} \frac{26n+37}{5} & n = 5p+13, p \geq 3 \\ \lceil \frac{26n+52}{5} \rceil - 4 & \text{otherwise.} \end{cases}$$

Table 1. Split domination numbers $\gamma_s(G_{m,n}), m, n \leq 25$

m /n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	2	21	22	23	24	25
1	-	-	-	-	-	-	-	-	-	-	–	-	-	–	-	-	-	-	-	–	-	-	-	-	-
2	-	2																							
3	1	2	3																						
4	2	3	4	5																					
5	2	4	5	6	8																				
6	2	4	6	7	9	11																			
7	3	5	7	8	10	12	13																		
8	3	5	7	9	12	13	15	17																	
9	3	6	8	10	13	15	17	19	21																
10	4	6	9	11	14	17	18	21	23	25															
11	4	7	10	12	15	18	20	23	25	28	30														
12	4	7	10	13	17	19	22	25	27	30	33	36													
13	5	8	11	14	18	21	23	27	30	32	36	39	41												
14	5	8	12	15	19	22	25	29	32	35	38	41	45	48											
15	5	9	13	16	20	23	27	30	34	37	41	44	48	51	54										
16	6	9	13	17	21	25	28	32	36	39	43	47	50	54	58	61									
17	6	10	14	18	23	27	30	34	38	42	46	50	54	57	61	65	69								
18	6	10	15	19	24	28	32	36	40	44	48	52	56	61	65	69	73	77							
19	7	11	16	20	25	29	33	38	42	46	51	55	59	64	68	72	76	81	85						
20	7	11	16	21	26	31	35	40	44	49	53	58	63	67	71	76	80	85	89	93					
21	7	12	17	22	27	32	37	42	46	51	56	61	65	70	75	79	84	89	93	98	102				
22	8	12	18	23	29	33	38	44	48	53	58	63	68	73	78	83	88	93	97	102	107	112			
23	8	13	19	24	30	35	40	45	50	55	61	66	71	76	81	87	92	97	102	107	112	117	122		
24	8	13	19	25	31	37	42	47	53	58	64	69	74	80	85	90	95	101	106	111	116	121	127	132	
25	9	14	20	26	32	38	43	49	55	60	66	72	77	83	88	94	99	105	110	115	121	126	132	137	142

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