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On the subdivided thorn graph and its metric dimension

Lyra Yulianti, Narwen, Sri Hariyani

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Andalas University, Padang, Indonesia

lyra@sci.unand.ac.id, narwen68@gmail.com, srihariyani@gmail.com

Abstract

For some ordered subset $W = \{w_1, w_2, \cdots, w_t\}$ of vertices in connected graph G, and for some vertex v in G, the metric representation of v with respect to W is defined as the t-vector $r(v|W) = \{d(v, w_1), d(v, w_2), \cdots, d(v, w_t)\}$. The set W is the resolving set of G if for every two vertices u, v in G, $r(u|W) \neq r(v|W)$. The metric dimension of G, denoted by $\dim(G)$, is defined as the minimum cardinality of W. Let G be a connected graph on n vertices. The thorn graph of G, denoted by $Th(G, l_1, l_2, \cdots, l_n)$, is constructed from G by adding l_i leaves to vertex v_i of G, for $l_i \geq 1$ and $1 \leq i \leq n$. The subdivided-thorn graph, denoted by $TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$, is constructed by subdividing every l_i leaves of the thorn graph of G into a path on y_i vertices. In this paper the metric dimension of thorn of complete graph, $\dim(Th(K_n, l_1, l_2, \cdots, l_n))$, $l_i \geq 1$ are determined, partially answering the problem proposed by Iswadi et al [7]. This paper also gives some conjectures for the lower bound of $\dim(Th(G, l_1, l_2, \cdots, l_n))$, for arbitrary connected graph G. Next, the metric dimension of subdivided-thorn of complete graph, $\dim(TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ are determined and some conjectures for the lower bound of $\dim(Th(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ for arbitrary connected graph G are given.

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1. Introduction

All graphs are considered finite, simple and undirected. Let G be an arbitrary connected graph on n vertices and $W = \{w_1, w_2, \cdots, w_t\}$ be an ordered subset of vertices in G. For some vertex v in G, the metric representation of v with respect to W is defined as t-vector $r(v|W) = \{d(v, w_1), d(v, w_2), \cdots, d(v, w_t)\}$. The set W is called the resolving set of G if for every two vertices u, v in G, $r(u|W) \neq r(v|W)$. The metric dimension of G, denoted by $\dim(G)$, is defined as the minimum cardinality of the resolving set W [2]. Other definitions and graph terminologies are taken from [3].

Chartrand et al. [2] gave the characterizations of some connected graph G with $\dim(G) = 1$, $\dim(G) = n - 1$ or $\dim(G) = n - 2$. They also gave the metric dimension of cycle C_n, arbitrary tree T, and the bounds for metric dimension of unicyclic graphs. There are many significant results related to the determination of the metric dimension of graphs. For example, the metric dimension of fan F_n [1], wheels W_n [9], n-partite complete graphs [8] and Jahangir graph [10].

For some connected graph G on n vertices, the thorn of G, denoted by $Th(G, l_1, l_2, \cdots, l_n)$, is a graph constructed by adding l_i leaves to vertex v_i of G, for $l_i \geq 1$ and $1 \leq i \leq n$ [11]. Iswadi et al [7] stated the metric dimension of $Th(G, l_1, l_2, \cdots, l_n)$ for $l_i \geq 2$ and gave the open problem for $l_i \geq 1$. That open problem are partially answered in this paper by determining the metric dimension of $Th(K_n, l_1, l_2, \cdots, l_n)$ for $l_i \geq 1$. This paper also gives some conjectures for the lower bound of $\dim(Th(G, l_1, l_2, \cdots, l_n))$, for arbitrary connected G and $l_i \geq 1$.

The subdivided-thorn graph, denoted by $TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$, are constructed by subdividing every l_i leaves of the thorn graph into a path on y_i vertices. Next, the metric dimension of $TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ for $l_i, y_i \geq 1$ and $1 \leq i \leq n$ are determined, and some conjectures for the lower bound of $\dim(TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ for $l_i, y_i \geq 1$ for arbitrary connected G are given in this paper.

2. Metric Dimension of Thorn of Complete Graph

In Theorem 2.1 the metric dimension of the thorn of complete graph, denoted by $H_1 \simeq Th(K_n, l_1, l_2, \cdots, l_n)$ for $l_i = 1,\ 1 \leq i \leq n$ is determined.

Theorem 2.1. *Let* $H_1 \simeq Th(K_n, 1, 1, \cdots, 1)$ *for* $1 \leq i \leq n$. *Then* $\dim(H_1) = n - 1$.

Proof. Let $H_1 \simeq Th(K_n, 1, 1, \cdots, 1)$, with $V(H_1) = \{v_i, v_{i1} | 1 \leq i \leq n\}$ and $E(H_1) = E(K_n) \cup \{v_i v_{i1}\}$. Let $W_1 = \{v_1, v_2, \cdots, v_{n-1}\}$. It will be proved that W_1 is the resolving set for H_1. Note that

1. $d(v_i, v_i) = 0, d(v_i, v_j) \neq 0, d(v_{i1}, v_i) = 1$ for $1 \leq i, j \leq n$.
2. $d(v_{i1}, v_i) = 1$ while $d(v_{j1}, v_i) = 2$ for $1 \leq i, j \leq n,\ i \neq j$.

Therefore, since $r(u|W_1) \neq r(v|W_1)$ for every two distinct vertices $u, v \in V(H_1)$, then W_1 is the resolving set for H_1. It is clear that $\dim(H_1) \leq n - 1$.

Next, it will be proved that W_1 is the minimum resolving set. Let $W_{11} = W_1 \setminus \{v_1\}$ be the resolving set, such that $|W_{11}| = n - 2$. For other possibilities, the proofs are similar. Because $d(v_1, v_t) = d(v_n, v_t)$ for every $t \in \{2, 3, \cdots, n - 2\}$, it can be easily obtained that $r(v_1|W_{11}) = r(v_n|W_{11})$. Therefore, W_{11} is not the resolving set. $\square$

The metric dimension of thorn of complete graph, denoted by $H_2 \simeq Th(K_n, l_1, l_2, \cdots, l_n)$, if $l_i = 1$, for some i, $1 \leq i \leq n$ is given in Theorem 2.2.

Theorem 2.2. *Let $H_2 \simeq Th(K_n, l_1, \cdots, l_n)$ with exactly one vertex, namely v_i, with $l_i = 1$ and $l_j \geq 2$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Then* $\dim(H_2) = \sum_{i=1}^{n}(l_i - 1)$.

Proof. Without loss of generality, let $l_1 = 1$ and $l_s \geq 2$ for $s \in \{2, 3, \cdots, n\}$. Let $W_2 = \{v_{st} | 2 \leq s \leq n, 1 \leq t \leq l_s - 1\}$, therefore $|W_2| = \sum_{s=2}^{n}(l_s - 1) = \sum_{i=1}^{n}(l_i - 1)$. It will be proved that W_2 is the resolving set for H_2. Note that

1. $d(v_s, v_{st}) = 1$, $d(v_i, v_{st}) = 2$ for $2 \leq s \leq n, 1 \leq t \leq l_s - 1$, and $1 \leq i \leq n, i \neq s$.
2. $d(v_{st}, v_{st}) = 0$, $d(v_{ij}, v_{st}) \neq 0$ for $2 \leq s \leq n, 1 \leq j, t \leq l_s - 1$, and $1 \leq i \leq n, i \neq s, j \neq t$.

Since $r(u|W_2) \neq r(v|W_2)$ for every two distinct vertices $u, v \in V(H_2)$, then W_2 is the resolving set for H_2. Therefore, $\dim(H_2) \leq \sum_{i=1}^{n}(l_i - 1)$.

It will be proved that W_2 is the minimum resolving set. Let $W_{21} = W_2 \setminus \{v_{21}\}$ be the resolving set, such that $|W_{21}| = n - 2$. For other possibilities, the proofs are similar. Because $d(v_1, v_t) = d(v_n, v_t)$ for every $t \in \{2, 3, \cdots, n-2\}$, then $r(v_1|W_{11}) = r(v_n|W_{11})$. Therefore, W_{11} is not the resolving set. Using the similar argument with the proof of Theorem 2.1, we have that $\dim(H_2) \geq \sum_{i=1}^{n}(l_i - 1)$. □

In Theorem 2.3 the metric dimension of $Th(K_n, l_1, l_2, \cdots, l_n)$, with $l_1 = l_2 = \cdots = l_k = 1$ for some k, $2 \leq k \leq n - 1$, $l_j \geq 2$ for $k + 1 \leq j \leq n$ is given.

Theorem 2.3. *Let $H_3 \simeq Th(K_n, l_1, \cdots, l_n)$, with $l_1 = l_2 = \cdots l_k = 1$ for k, $2 \leq k \leq n - 1$ and $l_j \geq 2$, for $k + 1 \leq j \leq n$. Then* $\dim(H_3) = \sum_{j=k+1}^{n}(l_j - 1) + (k - 1)$.

Proof. Let $H_3 \simeq Th(K_n, l_1, l_2, \cdots, l_n)$, with $l_1 = l_2 = \cdots = l_k = 1$, for $2 \leq k \leq n - 1$. Let $W_3 = \{v_s | 1 \leq s \leq k - 1\} \cup \{v_{jq} | k + 1 \leq j \leq n, 1 \leq q \leq l_j - 1\}$. Therefore, $|W_3| = \sum_{j=k+1}^{n}(l_j - 1) + (k - 1)$. It will be shown that W_3 is the resolving set. Note that

1. $d(v_s, v_s) = 0$, $d(v_i, v_s) \neq 0$ for $1 \leq s \leq k - 1, 1 \leq i \leq n, i \neq s$.
2. $d(v_{jq}, v_{jq}) = 0$, $d(v_{mt}, v_{jq}) \neq 0$ for $k + 1 \leq j, m \leq n, 1 \leq q, t \leq l_j - 1$, and $m \neq j, q \neq t$.

Therefore, since $r(u|W_3) \neq r(v|W_3)$ for every two distinct vertices $u, v \in V(H_3)$, W_3 is the resolving set for H_3. Then $\dim(H_3) \leq \sum_{j=k+1}^{n}(l_j - 1) + (k - 1)$.

Using the similar argument with the proof of Theorem 2.1, we have $\dim(H_3) \geq \sum_{j=k+1}^{n}(l_j - 1) + (k - 1)$. □

Based on Theorem 2.1 – Theorem 2.3, Conjecture 2.1 – 2.3 give the lower bounds of metric dimension of thorn of arbitrary connected graph G on n vertices.

Conjecture 2.1. *Let $H_4 \simeq Th(G, l_1, \cdots, l_n)$ with exactly one vertex, namely v_i with $l_i = 1$, $l_j \geq 2$ for $i, j \in \{1, 2, \cdots, n\}$, $i \neq j$. Then* $\dim(H_4) \geq \sum_{i=1}^{n}(l_i - 1)$.

Conjecture 2.2. *Let $H_5 \simeq Th(G, l_1, l_2, \cdots, l_n)$ with $l_1 = l_2 = \cdots = l_n = 1$. Then* $\dim(H_5) \geq n - 1$.

Conjecture 2.3. *Let $H_6 \simeq Th(G, l_1, \cdots, l_n)$ with $l_1 = l_2 = \cdots l_k = 1$ for some k, $2 \leq k \leq n - 1$ and $l_j \geq 2$, for $k + 1 \leq j \leq n$. Then* $\dim(H_6) \geq \sum_{j=k+1}^{n}(l_j - 1) + (k - 1)$.

3. Metric Dimension of Subdivided-thorn Graph

The subdivided-thorn graph $TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ is constructed by subdividing every l_i leaves of the thorn graph $Th(G, l_1, l_2, \cdots, l_n)$, into a path on y_i vertices, where $l_i, y_i \geq 1$, $1 \leq i \leq n$. The subdivided-thorn of arbitrary connected G is given in Figure 1.

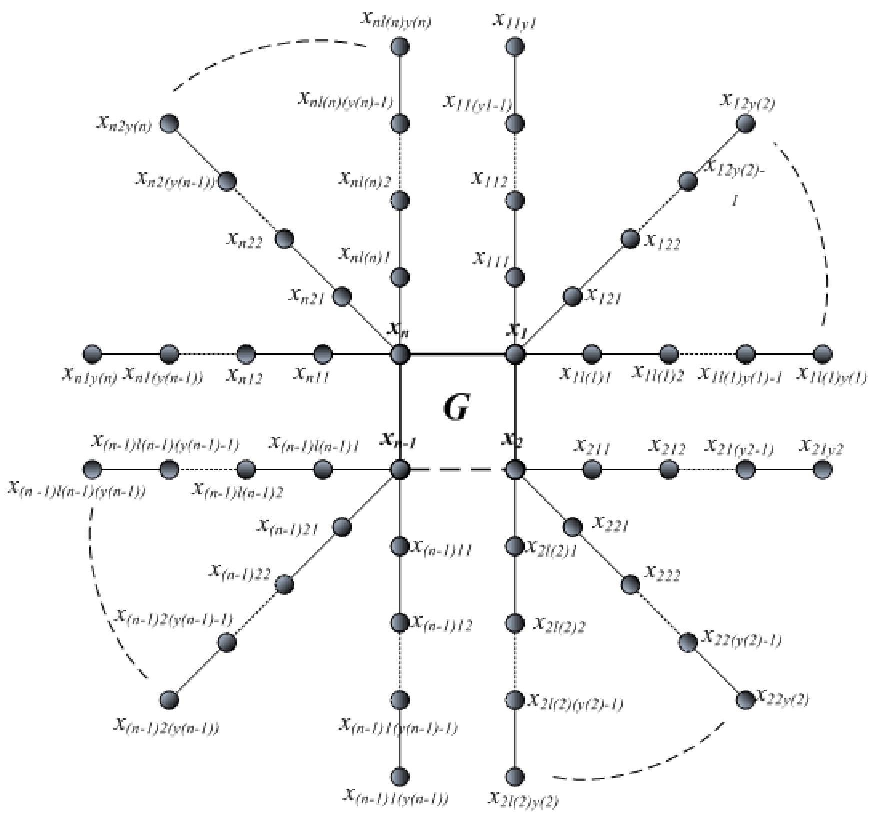

Figure 1. $TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$

The metric dimensions of subdivided-thorn graph of complete graph K_n are given in Theorem 3.1 – Theorem 3.3.

Theorem 3.1. *Let* $L_1 \simeq TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$, *where* $l_i = l$ *and* $y_i \geq 1$ *for* $1 \leq i \leq n$. *Then* $\dim(L_1) = n - 1$.

Proof. Let $L_1 \simeq TD(K_n, 1(y_1), 1(y_2), \cdots, 1(y_n))$, with $V(L_1) = \{v_i, v_{i1k} \mid 1 \leq i \leq n, 1 \leq k \leq y_i\}$ and $E(L_1) = E(K_n) \cup \{v_i v_{i1k} \mid 1 \leq i \leq n, 1 \leq k \leq y_i\}$. Let $Y_1 = \{v_{11y_1}, v_{21y_2}, \cdots, v_{(n-1)1y_{n-1}}\}$. It will be proved that Y_1 is the resolving set for L_1. Note that

1. $d(v_j, v_{j1y_j}) = y_j$, $d(v_t, v_{j1y_j}) = y_j + 1$, and $d(v_t, v_{t1y_t}) = y_t$, $d(v_j, v_{t1y_t}) = y_t + 1$, for $1 \leq j, t \leq n - 1$, and $j \neq t$; thus the representation of v_j and v_t are different in the i^{th} and the j^{th} position.
2. $d(v_{11y_1}, v_{11y_1}) = 0$, $d(v_{i1k}, v_{11y_1}) \neq 0$ for $1 \leq i \leq n$, $1 \leq k \leq y_i$.

Therefore, since $r(u|Y_1) \neq r(v|Y_1)$ for every two distinct vertices $u, v \in V(L_1)$, Y_1 is the resolving set for L_1. Then, $\dim(L_1) \leq n-1$.

Next, it will be shown that Y_1 is the minimum resolving set for L_1. Let $Y_{11} = Y_1 \setminus \{v_{111}\}$ be the resolving set, such that $|Y_{11}| = n-2$. Because $d(v_1, v_{j1y_j}) = d(v_n, v_{j1y_j})$ for every $j \in \{2, 3, \cdots, n-2\}$, then $r(v_1|Y_{11}) = r(v_n|Y_{11})$. Therefore, Y_{11} is not the resolving set. $\square$

The metric dimension of $TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n)$, with $l_i = 1$ for some i, $1 \leq i \leq n$ is given in Theorem 3.2.

Theorem 3.2. *Let $L_2 \simeq TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ with exactly one vertex, namely v_i with $l_i = 1$ and $l_j \geq 2$ for $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Then* $\dim(L_2) = \sum_{i=1}^{n}(l_i - 1)$.

Proof. Without loss of generality, let $l_1 = 1$ and $l_s \geq 2$ for $s \in \{2, 3, \cdots, n\}$. Let $Y_2 = \{v_{sty_s}|2 \leq s \leq n, 1 \leq t \leq l_s - 1\}$, therefore $|Y_2| = \sum_{s=2}^{n}(l_s - 1) = \sum_{i=1}^{n}(l_i - 1)$. It will be shown that Y_2 is the resolving set for L_2. Note that

1. $d(v_s, v_{sty_s}) = y_s$, $d(v_i, v_{sty_s}) = y_s + 1$, for $2 \leq s \leq n, 1 \leq t \leq l_s - 1, 1 \leq i \leq n, i \neq s$.
2. $d(v_{sty_s}, v_{sty_s}) = 0$, $d(v_{ijk}, v_{sty_s}) \neq 0$, for $2 \leq s \leq n, 1 \leq t \leq l_s - 1, 1 \leq i \leq n, 1 \leq j \neq l_i$.

Therefore, since $r(u|Y_2) \neq r(v|Y_2)$ for every two distinct vertices $u, v \in V(L_2)$, Y_2 is the resolving set for L_2. Then $\dim(L_2) \leq \sum_{i=1}^{n}(l_i - 1)$.

By using the similar argument with the proof of Theorem 3.1, then we have $\dim(L_2) \geq \sum_{i=1}^{n}(l_i - 1)$. $\square$

In Theorem 3.3 the metric dimension of $TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$, with $l_1 = l_2 = \cdots l_k = 1$ for some k, $2 \leq k \leq n-1$ and $l_j \geq 2$, for $k+1 \leq j \leq n$ is given.

Theorem 3.3. *Let $L_3 \simeq TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ with $l_1 = l_2 = \cdots l_k = 1$ for some k, $2 \leq k \leq n-1$ and $l_j \geq 2$, for $k+1 \leq j \leq n$. Then* $\dim(L_3) = \sum_{j=k+1}^{n}(l_j - 1) + (k-1)$.

Proof. Let $L_3 \simeq TD(K_n, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ with $l_1 = l_2 = \cdots l_k = 1$ for some k, $2 \leq k \leq n-1$ and $l_j \geq 2$, for $k+1 \leq j \leq n$. Let $Y_3 = \{v_s|1 \leq s \leq k-1\} \cup \{v_{jqy_j}|k+1 \leq j \leq n, 1 \leq q \leq l_j - 1\}$. Then, $|Y_3| = \sum_{j=k+1}^{n}(l_j - 1) + (k-1)$. It will be shown that Y_3 is the resolving set for L_3. Note that

1. $d(v_s, v_s) = 0$, $d(v_i, v_s) \neq 0$ for $1 \leq s \leq k-1, 1 \leq i \leq n, i \neq s$.
2. $d(v_{jqy_j}, v_{jqy_j}) = 0$, $d(v_{mtp}, v_{jqy_j}) \neq 0$ for $k+1 \leq j, m \leq n, 1 \leq q, t \leq l_j - 1, p \neq y_j$, and $m \neq j, q \neq t$.

Therefore, since $r(u|Y_3) \neq r(v|Y_3)$ for every two distinct vertices $u, v \in V(L_3)$, Y_3 is the resolving set for L_3. Then, $\dim(L_3) \leq \sum_{j=k+1}^{n}(l_j - 1) + (k-1)$.

By using the similar argument with the proof of Theorem 3.1, we have that $\dim(L_3) \geq \sum_{j=k+1}^{n}(l_j - 1) + (k-1)$. $\square$

Based on Theorem 3.1 – Teorema 3.3, some conjectures on the lower bound for metric dimension of the thorn-subdivided graph for arbitrary connected graph G are given.

Conjecture 3.1. *Let* $L_4 \simeq TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ *with exactly one vertex, namely* v_i *with* $l_i = 1$ *and* $l_j \geq 2$ *for* $i, j \in \{1, 2, \cdots, n\}$. *Then* $\dim(L_4) \geq \sum_{i=1}^{n}(l_i - 1)$.

Conjecture 3.2. *Let* $L_5 \simeq TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ *with* $l_1 = l_2 = \cdots = l_n = 1$. *Then* $\dim(L_5) \geq n - 1$.

Conjecture 3.3. *Let* $L_6 \simeq TD(G, l_1(y_1), l_2(y_2), \cdots, l_n(y_n))$ *with* $l_1 = l_2 = \cdots l_k = 1$ *for some* k, $2 \leq k \leq n-1$ *and* $l_j \geq 2$, *for* $k+1 \leq j \leq n$. *Then* $\dim(L_6) \geq \sum_{j=k+1}^{n}(l_j - 1) + (k-1)$.

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