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Edge magic total labeling of lexicographic product $C_{4(2r+1)} \circ \overline{K_2}$, cycle with chords, unions of paths, and unions of cycles and paths

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Abstract

An *edge magic total (EMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to a set of numbers defined by $\lambda : V \cup E \to \{1, 2, ..., |V| + |E|\}$ with the property that for every $xy \in E$, the weight of xy equals to a constant k, that is, $\lambda(x) + \lambda(y) + \lambda(xy) = k$ for some integer k. In this paper given the construction of an EMT labeling for certain lexicographic product $C_{4(2r+1)} \circ \overline{K_2}$, cycle with chords ${}^{[c]t}C_n$, unions of paths mP_n, and unions of cycles and paths $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$.

Keywords: edge magic total labeling, lexicographic product, cycle with chords, unions of paths, unions of cycles and paths
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1. Introduction

Let G be a simple graph with vertex set V and edge set E. A *total labeling* of G is a bijection $f\colon V \cup E \to \{1, 2, \dots, |V| + |E|\}$. If $x, y \in V$ and if $e = xy \in E$, then the weight $w(e)$ of the edge e is given by $w(e) = f(x) + f(y) + f(e)$. The total labeling f is said to be an *edge-magic total (EMT) labeling* if the weight of each edge is a constant, and this constant is called the *magic constant* of the EMT labeling. An EMT labeling is called *super edge-magic total (SEMT) labeling* if the vertices are labeled using the smallest $|V|$ integers.

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In [8] Kotzig and Rosa stated the result for an EMT labeling of a complete bipartite graph. They used the terminology M-valuation, which is now known as EMT labeling. They also stated the preservation an EMT labeling for odd number of copies of certain graphs. They use the term *edge-magic* to describe graph that has an EMT labeling.

Theorem 1. [8] *An M-valuation [EMT labeling] of the complete bipartite graph $K_{p,q}$ exists for all $p, q \geq 1$.*

Theorem 2. [8] *Say G is a 3-colorable edge-magic graph and H is the union of t disjoint copies of G, t odd. Then H is edge magic.*

In [11], a method to expand the result in EMT labeling for some families of graphs was introduced in an MS thesis. In [1], this method is used to generalize the results for EMT labeling of 2-regular graphs. In this paper we apply the method to construct an EMT labeling for several other families of graphs. In Section 2 the main theorem consisting the expansion method is given. The following sections gives the new results for each families of graphs, along with some examples that describes how the method works.

Extending Cycles and Copying Paths

In this section we will describe first the method that later applied to construct an EMT labelings. This method preserves the EMT (SEMT) properties as we extend the length of cycles, or multiplying the number of path, by a factor of an odd number.

In [10] Marr and Wallis give a definition of a Kotzig array as a $d \times m$ grid, each row being a permutation of $\{0, 1, \ldots, m-1\}$ and each column having the same sum. The Kotzig array used in this paper is the $3 \times (2r+1)$ Kotzig array κ that is given as an example in [10] after adding each entry of the array by one:

$$\kappa = \begin{bmatrix} 1 & 2 & \ldots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ r+1 & r+2 & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 2r+1 & 2r-1 & \ldots & 1 & 2r & \ldots & 4 & 2 \end{bmatrix}$$

If we write the first two rows of κ as a permutation cycle τ, we have:

$$\tau = (1, r+1, 2r+1, r, 2r, \ldots, 3, r+3, 2, r+2)$$

The difference between two consecutive elements in τ is equal to r taken modulo $(2r+1)$. Note that τ is a $(2r+1)$-cycle. Since $(2r+1)$ is an odd number for every nonnegative integer r, then $\gcd(2, 2r+1) = 1$, and so we have τ^2 also a $(2r+1)$-cycle. This fact plays an important role in preserving the properties of magic labeling of our EMT and SEMT labeling as we extend the length of cycles.

Let κ' be the modified κ, where we switched the first and second row of κ:

$$\kappa' = \begin{bmatrix} r+1 & r+2 & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 1 & 2 & \ldots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ 2r+1 & 2r-1 & \ldots & 1 & 2r & \ldots & 4 & 2 \end{bmatrix}$$

It is clear that if we write the first two rows of κ' as a permutation cycle, we have τ^{-1}.

Theorem 3. *Let G be a 2-regular graph that has an EMT labeling μ. Let G' be a 2-regular graph obtained by extending the length of each component of G by an odd factor. Then there exists an EMT labeling for G' that can be obtained by modifying the EMT labeling of G.*

Proof. Let μ be an EMT labeling for any 2-regular graph G. For every vertex and edge of G, let λ be the labeling obtained by decreasing the original label by 1, that is, let $\lambda(v) = \mu(v) - 1$ and $\lambda(e) = \mu(e) - 1$.
For each cycle C_n in G, construct an $n \times 3$ table with entries as follows.

- In the first column: For $i = 1, 2, ..., n$, the entry in the i^{th} row is the 3×1 matrix $\Lambda = \begin{bmatrix} \lambda(v_i) \\ \lambda(v_{i+1}) \\ \lambda(e_{i+1}) \end{bmatrix}$.
- In the second column: For $y = 1, 2, 3$ and $z = 1, 2, ..., (2r + 1)$ the entry in the i^{th} row is either κ or κ' depending on the value of i, namely $\kappa = [\kappa_{yz}]$, if $i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, and $\kappa' = [\kappa'_{yz}]$, if $\left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n$, where κ_{yz} denotes the element on the y^{th} row and z^{th} column of κ.
- In the third column: For $i = 1, 2, ..., n$, the entry in the i^{th} row is the matrix $\Theta_i = \begin{cases} [\kappa_{yz} + (2r+1)\Lambda_{y1}], & \text{if } i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ [\kappa'_{yz} + (2r+1)\Lambda_{y1}], & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n \end{cases}$

If we multiply the permutation cycles of κ and κ' in the second column, we obtain

$$\tau^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \tau^{n - \left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)} = \tau^{2\left\lfloor \frac{n}{2} \right\rfloor - n + 2}$$

If n is odd we have $\tau^{(n-1)-n+2} = \tau$ and if n is even we have $\tau^{n-n+2} = \tau^2$.
The cycle $C_{n(2r+1)}$ is obtained by tracking the numbers on Θ. Let θ^i_{yz} denote the element of Θ_i in the y^{th} row and z^{th} column. In each Θ_i, the two numbers θ^i_{1z} and θ^i_{2z} will be the labels of two adjacent vertices on $C_{n(2r+1)}$, and θ^i_{3z} will be the label of the edge they share.
For each $i, 1 \leq i \leq n$, each pair of $\left\{\theta^{i+1}_{1z}\right\}$ and $\left\{\theta^i_{2z}\right\}$ that are equal denotes the same vertex on $C_{n(2r+1)}$ and all pairs θ^i_{1z} and θ^i_{2z} represent labels of adjacent vertices.
Recall that in the second column, τ is a permutation cycle of length $2r + 1$. Both 1 and 2 are relatively prime to $2r + 1$ for any integer r, so $\tau = \tau^1$ and τ^2 are also permutation cycles of length $2r+1$. Consequently, we can track the labeling of $C_n(2r+1)$ by connecting these vertices from the third column continuously until we get a full circle of longer length (not stopping until all numbers in the third column are used). Since $1 \leq z \leq 2r + 1$, the result from this process is the labeled extended cycle $C_n(2r + 1)$.
For path component of G we create the same table, but since there is no relation between the endpoints, when tracking adjacent vertices in Θ_i from $i = 1$ until $i = m$, we will not be able to go back to $i = 1$. Every time we track adjacent vertices from $i = 1$ until $i = m$, we will get one copy of P_m instead. Since we have $(2r + 1)$ columns in each Θ_i, we end up with $(2r + 1)$ copies of P_m instead of $P_{m(2r+1)}$. Combining all extended components, we obtain an EMT labeling for G'. □

2. Results

Lexicographic Product $C_{4(2r+1)} \circ \overline{K_2}$

The *lexicographic product* or *graph composition* $G \circ H$ or $G[H]$ of graphs G and H is a graph such that the vertex set of $G \circ H$ is the Cartesian product $V(G) \,\square\, V(H)$, and any two vertices (u, v) and (x, y) are adjacent in $G \circ H$ if and only if either u is adjacent to x in G or $u = x$ and v is adjacent to y in H.
In this section we apply Theorem 3 to construct an EMT labeling of $C_{4(2r+1)} \circ \overline{K_2}$ from an EMT labeling of $C_4 \circ \overline{K_2}$.
Observe that $K_{4,4} \cong C_4 \circ \overline{K_2}$, so an EMT labeling for $C_4 \circ \overline{K_2}$ exists. An EMT labeling for the complete bipartite graph $K_{4,4}$ as given in [8] is shown in Figure 1.

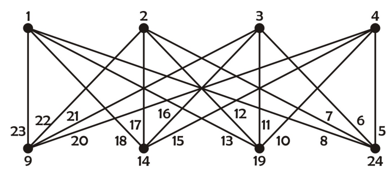

Figure 1. EMT labeling for $K_{4,4}$

Figure 1 is isomorphic with the lexicographic product $C_4 \circ \overline{K_2}$ shown in the Figure 2.

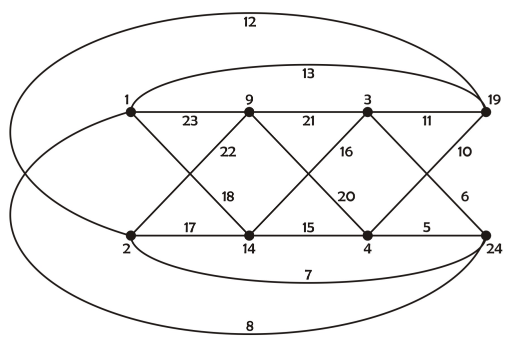

Figure 2. EMT labeling for $C_4 \circ \overline{K_2}$

Example: $C_4 \circ \overline{K_2} \to C_{12} \circ \overline{K_2}$
In this example we will show how to use Theorem 3 to find an EMT labeling of $C_{4(2r+1)} \circ \overline{K_2}$ from the known EMT labeling of $C_4 \circ \overline{K_2}$.
First decompose the graph $C_4 \circ \overline{K_2}$ into $4C_4$, where each edge in the edge set of $C_4 \circ \overline{K_2}$ is used exactly once, as shown in Figure 3. Apply Theorem 3 separately for each cycles.
We expand using the factor $(2r + 1) = 3$ to get $4C_{12}$. Construct a table for each cycle as defined in the proof of Theorem 3: The original cycles are $C_n = C_4$, so the first $\left\lfloor \frac{n}{2} \right\rfloor + 1 = 3$ rows use the array κ, while the fourth row use κ'. The tables are shown in Table 1. The entries of Θ_i then

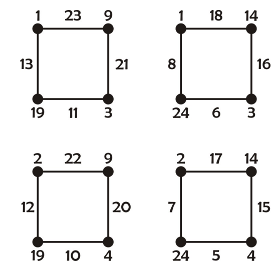

Figure 3. Decomposed $C_4 \circ \overline{K_2}$

obtained by multiplying entry of Λ with the factor $(2r+1)$, then added by the corresponding entry of κ or κ', e.g., in the second row of Θ_1 of the upper left table we have the entry $8 \times 3 + 2 = 26$.

Λ	κ or κ'			Θ_i		
0	1	2	3	1	2	3
8	2	3	1	26	27	25
22	3	1	2	69	67	68
8	1	2	3	25	26	27
2	2	3	1	8	9	7
20	3	1	2	63	61	62
2	1	2	3	7	8	9
18	2	3	1	56	57	55
10	3	1	2	33	31	32
18	2	3	1	56	57	55
0	1	2	3	1	2	3
12	3	1	2	39	37	38

Λ	κ or κ'			Θ_i		
0	1	2	3	1	2	3
13	2	3	1	41	42	40
17	3	1	2	54	52	53
13	1	2	3	40	41	42
2	2	3	1	8	9	7
15	3	1	2	48	46	47
2	1	2	3	7	8	9
23	2	3	1	71	72	70
5	3	1	2	18	16	17
23	2	3	1	71	72	70
0	1	2	3	1	2	3
7	3	1	2	24	22	23

Λ	κ or κ'			Θ_i		
1	1	2	3	4	5	6
8	2	3	1	26	27	25
21	3	1	2	66	64	65
8	1	2	3	25	26	27
3	2	3	1	11	12	10
19	3	1	2	60	58	59
3	1	2	3	10	11	12
18	2	3	1	56	57	55
9	3	1	2	30	28	29
18	2	3	1	56	57	55
1	1	2	3	4	5	6
11	3	1	2	36	34	35

Λ	κ or κ'			Θ_i		
1	1	2	3	4	5	6
13	2	3	1	41	42	40
16	3	1	2	51	49	50
13	1	2	3	40	41	42
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
23	2	3	1	71	72	70
4	3	1	2	15	13	14
23	2	3	1	71	72	70
1	1	2	3	4	5	6
6	3	1	2	21	19	20

Table 1. Tables for $C_4 \circ \overline{K_2} \to C_{12} \circ \overline{K_2}$

From each table we construct on C_{12}. In the upper left table, the first column of Θ_1 implies that 1 and 26 are the labels of adjacent vertices and the label of the edge they share is 69. The second column of Θ_2 implies that 26 and 9 are the labels of adjacent vertices and the label of the edge they share is 61. Continuing in a similar way for $\Theta_i, 1 \leq i \leq 4$ we get a path with 4 edges, with $(1, 26, 9, 55, 3)$ as the label of its vertices, in consecutive order.

Going back from Θ_4 to the third column of Θ_1, we see that 3 and 25 are labels of adjacent vertices and the label of the edge they share is 68, and so on. Continuing until all entries in every Θ_i are used, we get a cycle of length 12 in which the vertices are labeled in consecutive order with $(1, 26, 9, 55, 3, 25, 8, 57, 2, 27, 7, 56)$. Repeat similar steps for the remaining tables to obtain a EMT labeling for $4C_{12}$ with $k = 96$ as shown in Figure 4. Then recombine these cycles to get $C_{12} \circ \overline{K_2}$ as shown in Figure 5.

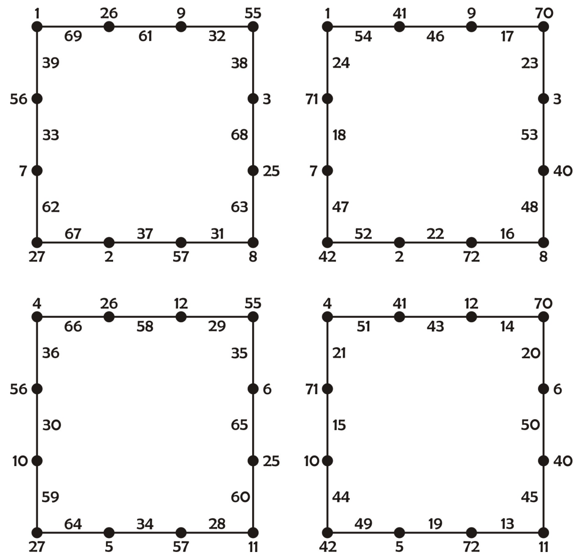

Figure 4. Extended decomposed $C_4 \circ \overline{K_2}$

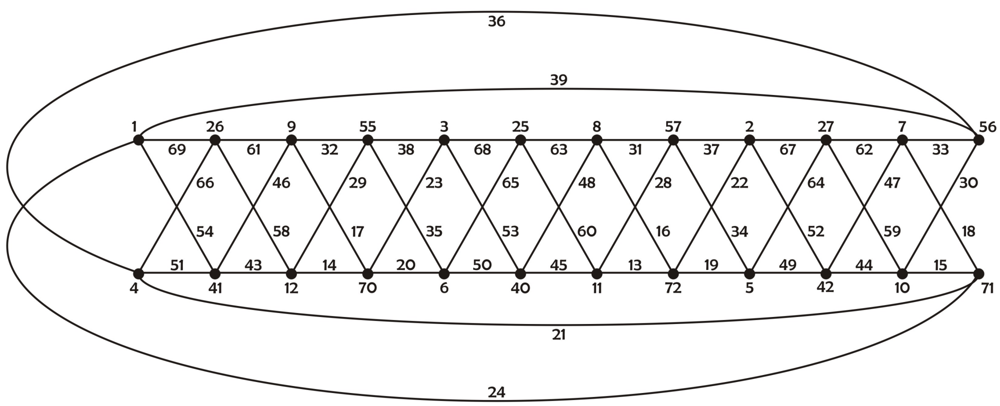

Figure 5. EMT labeling for $C_{12} \circ \overline{K_2}$

Theorem 4. *For $r \geq 0$, odd values of m, the graph $m\left(C_{4(2r+1)} \circ \overline{K_2}\right)$ has an EMT labeling.*

Proof. Applying Theorem 3 to Theorem 1, we get an EMT labeling of $C_{4(2r+1)} \circ \overline{K_2}$ for any r. Applying Theorem 2 to this labeling we get an EMT labeling of $m\left(C_{4(2r+1)} \circ \overline{K_2}\right)$ for an odd value of m. □

Cycles with Chords ${}^tC_n$

In this paper ${}^tC_n$ denotes a cycle C_n with one chord of length t, while ${}^{[c]t}C_n$ denotes a cycle C_n with c chords, each of length t. In [9] several results for SEMT of cycle with one chord are known and are stated in Theorem 5.

Theorem 5. [9] *The following cycles with one chord has an SEMT labeling:*

(a) ${}^tC_{4m+1}$ *for all* t *other than* $t = 5, 9, 4m-4, 4m-8$, *given* $m \geq 3$.

(b) ${}^tC_{4m+1}$ *for all* $t \equiv 1 \pmod 4$ *except* $t = 4m-3$.

(c) ${}^tC_{4m}$ *for any* m *and* $t \equiv 2 \pmod 4$.

(d) ${}^tC_{4m+2},\ m \geq 2$ *for* $t = 2, 6$ *and all odd* t *other than* 5.

We apply Theorem 2 and Theorem 3 to general cycles with one chord. The cycle part get extended, while the number of chords multiplied.

Theorem 6. *If the graph ${}^tC_n$ has an EMT (SEMT) labeling, then there exists positive integers h and t_h such that for every integer $r \geq 0$, the graph ${}^{[(2r+1)^h]t_h}C_{n(2r+1)^h}$ also has an EMT (SEMT) labeling.*

Proof. Applying Theorem 3 to an SEMT labeling of ${}^tC_n$ gives an SEMT labeling of ${}^{[(2r+1)]t_1}C_{n(2r+1)}$ for some value of $t_1 \in \mathbb{N}$. Applying Theorem 3 to the SEMT labeling of ${}^{[(2r+1)]t_1}C_{n(2r+1)}$ gives an SEMT labeling of ${}^{[(2r+1)^2]t_2}C_{n(2r+1)^2}$ for some value of $t_2 \in \mathbb{N}$. Performing this h times for any finite number h, we get an SEMT labeling for ${}^{[(2r+1)^h]t_h}C_{n(2r+1)^h}$. □

The pattern of how the length of the chord changes is still unknown. However, we do know the location of the chords. Suppose ${}^tC_n$ has a chord connecting the vertices labeled $\lambda(v_a)$ and $\lambda(v_b)$ where $v_a, v_b \in V$. Denote these vertices with the pair notation $(\lambda(v_a), \lambda(v_b))$. Applying Theorem 3 to ${}^tC_n$, we get ${}^{[2r+1]t_1}C_{(2r+1)n}$. The set of $(2r+1)$ chords written in the pair notation is $\left\{(2r+1)(\lambda(v_a)-1)+\kappa_{1j}, (2r+1)(\lambda(v_b)-1)+\kappa_{2j}\right\}$.

Since we can use different κ and κ' combination in the second column of the table, the number of expanded graph is not unique. If we arrange the second column in the table in such a manner that we have τ^a where $a \neq 1, a \neq 2$ and a is relatively prime to n, then we can obtain different expanded graphs.

In the following example we use both the proposed $\tau^4(\tau^{-1})^3 = \tau$ and the possible alternative $\tau^7(\tau^{-1})^0 = \tau^7$ for expanding ${}^2C_7$.

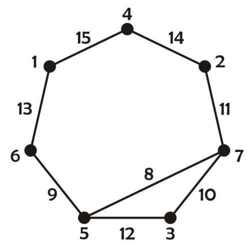

Figure 6. SEMT labeling for ${}^2C_7$

Example: ${}^2C_7 \to {}^{[5]9}C_{35}$ *and* ${}^2C_7 \to {}^{[3]5}C_{21}$
An SEMT labeling for ${}^2C_7$ with $k = 20$ as given in [9] is shown in Figure 6.
We expand using the factor $2r + 1 = 3$. The table for the chord is is given in Table 2.

Λ	κ			Θ_i		
6	1	2	3	19	20	21
4	2	3	1	14	15	13
7	3	1	2	24	22	23

Table 2. Table for the chord on both ${}^2C_7 \to {}^{[3]9}C_{21}$ and ${}^2C_7 \to {}^{[3]5}C_{21}$

The distinct tables using different κ and κ' combination for the cycle are given in Table 3. The second column of the left table is as defined in the proof of Theorem 3: The original cycle is $C_n = C_7$, so the first $\left\lfloor \frac{n}{2} \right\rfloor + 1 = 4$ rows use the array κ, while the other rows use the array κ'. The right table, on the other hand, use κ for every rows. Both tables works, but they gives different expanded graphs.

Λ	κ or κ'			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
1	2	3	1	5	6	4
13	3	1	2	42	40	41
1	1	2	3	4	5	6
6	2	3	1	20	21	19
10	3	1	2	33	31	32
6	1	2	3	19	20	21
2	2	3	1	8	9	7
9	3	1	2	30	28	29
2	2	3	1	8	9	7
4	1	2	3	13	14	15
11	3	1	2	36	34	35
4	2	3	1	14	15	13
5	1	2	3	16	17	18
8	3	1	2	27	25	26
5	2	3	1	17	18	16
0	1	2	3	1	2	3
12	3	1	2	39	37	38

Λ	κ or κ'			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
1	2	3	1	5	6	4
13	3	1	2	42	40	41
1	1	2	3	4	5	6
6	2	3	1	20	21	19
10	3	1	2	33	31	32
6	1	2	3	19	20	21
2	2	3	1	8	9	7
9	3	1	2	30	28	29
2	1	2	3	7	8	9
4	2	3	1	14	15	13
11	3	1	2	36	34	35
4	1	2	3	13	14	15
5	2	3	1	17	18	16
8	3	1	2	27	25	26
5	1	2	3	16	17	18
0	2	3	1	2	3	1
12	3	1	2	39	37	38

Table 3. Table for the cycle ${}^2C_7 \to {}^{[3]9}C_{21}$ (left) and ${}^2C_7 \to {}^{[3]5}C_{21}$ (right)

From the tables get an SEMT for ${}^{[3]9}C_{21}$ and for ${}^{[3]5}C_{21}$ with $k = 57$, as shown in Figure 7.

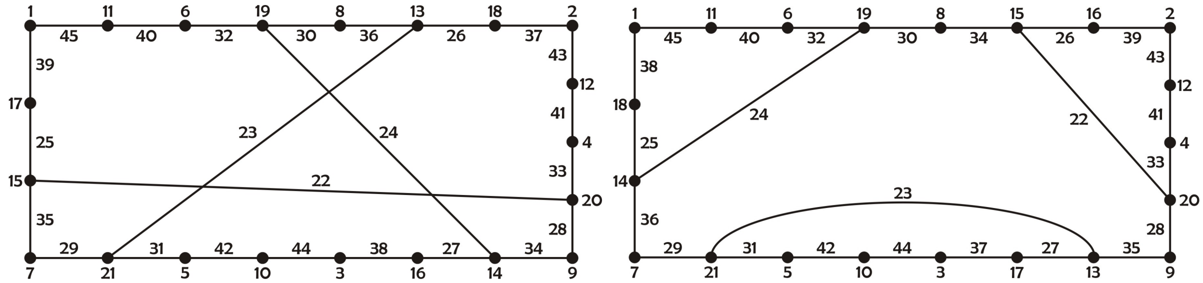

Figure 7. SEMT labeling for ${}^{[3]9}C_{21}$ (left) and ${}^{[3]5}C_{21}$ (right)

Unions of Paths mP_n

In [3], [5] and [6], several results for SEMT of unions of paths are known:

Theorem 7. *The following graph has an SEMT labeling:*

(a) [3] *The graph* $F \cong P_m \cup P_n$*, iff* $(m, n) \neq (2, 2)$ *or* $(3, 3)$.

(b) [5] mP_n*, if* m *is odd.*

(c) [5] $P_3 \cup mP_2$ *for all* m.

(d) [5] $m(P_2 \cup P_n)$*, if* m *is odd and* $n \in \{3, 4\}$.

(e) [6] $2P_n$ *iff* n *is not* 2 *or* 3.

(f) [6] $2P_{4n}$ *has an SEMT labeling for all* n.

Note that Theorem 7(e) is a special case of Theorem 7(a) when $m = n$. Applying Theorem 3 to Theorem 7 above we can summarize our new results in Theorem 8

Theorem 8. *The following graph has an SEMT labeling:*

(a) $(2r + 1)\,(P_m \cup P_n)$*, for any* r*, if* $(m, n) \neq (2, 2)$ *or* $(3, 3)$.

(b) $(2r + 1)\,(P_3 \cup mP_2)$ *for any* m *and* r.

(c) mP_n *for even values of* m*,* $m \equiv 2 \pmod 4$*, if* $n \neq 2, 3$.

(d) mP_{4n} *for even value of* m*,* $m \equiv 2 \pmod 4$ *and all* $n \geq 2$

Proof. Apply Theorem 3 to Theorem 7. □

Union of Cycles and Paths

In [3] several results for SEMT of unions of cycles and paths are known:

Theorem 9. **[3]** *The following graph has an SEMT labeling:*

(a) $C_3 \cup P_{n_2}$, *if* $n_2 \geq 6$.

(b) $C_4 \cup P_{n_2}$, *if* $n_2 \neq 3$.

(c) $C_5 \cup P_{n_2}$, *if* $n_2 \geq 4$.

(d) $C_{n_1} \cup P_{n_2}$, *if* n_1 *is even and* $n_2 \geq \frac{n_1}{2} + 2$.

Theorem 10. *The following graph has an SEMT labeling:*

(a) $m\left(C_{3(2r+1)} \cup (2r+1)P_{n_2}\right)$, *for any* $r \geq 0$, *odd* m, *and* $n_2 \geq 6$.

(b) $m\left(C_{4(2r+1)} \cup (2r+1)P_{n_2}\right)$, *for any* $r \geq 0$, *odd* m, *and* $n_2 \neq 3$.

(c) $m\left(C_{5(2r+1)} \cup (2r+1)P_{n_2}\right)$, *for any* $r \geq 0$, *odd* m, *and* $n_2 \geq 4$.

(d) $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$, *for any* $r \geq 0$, *odd* m, *even* $n_1 \geq 4$, *and* $n_2 \geq \frac{n_1}{2} + 2$.

Proof. Apply Theorem 2 and Theorem 3 to Theorem 9. □

Corollary 2.1. *For any non-negative integer* r, *odd* m *and any positive integer* n_2, *the graph* $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$ *has an SEMT labeling when* $n_1 = 4, 5, 6, 8$ *or* 10, *unless* $(n_1, n_2) = (4, 3), (6, 1), (10, 1)$.

Example: $C_4 \cup P_2 \rightarrow C_{20} \cup 5P_2$
An SEMT labeling for $C_4 \cup P_2$ as given in [3] is shown in Figure 8.

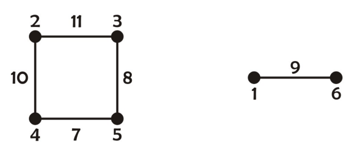

Figure 8. SEMT labeling for $C_4 \cup P_2$

We expand using the factor $2r + 1 = 5$. The tables are given in Table 4. For the table for $C_n = C_4$ (left table), the first $\left\lfloor \frac{n}{2} \right\rfloor + 1 = 3$ rows use the array κ, while the fourth row use the array κ'. The table for path P_2 (right table) use only κ since there is only one row.
Figure 9 shows the SEMT labeling for $C_{20} \cup 5P_2$ with $k = 74$.

More Results

Theorem 3 is in fact applicable to any graph that has EMT or SEMT labeling. The expanded graphs, however, either overlapped with results that already known, or might have little to none regularity that is of interest in magic labeling.

Applying Theorem 3 to an SEMT labeling Cartesian product $P_m \square C_n$ (*prism* graph) gives an SEMT labeling of $P_m \square C_{n(2r+1)}$. However, all results for SEMT labelings of $P_m \square C_n$ for odd values of n are already given in [4].

Λ	κ or κ'					Θ_i				
1	1	2	3	4	5	6	7	8	9	10
2	3	4	5	1	2	13	14	15	11	12
10	5	3	1	4	2	55	53	51	54	52
2	1	2	3	4	5	11	12	13	14	15
4	3	4	5	1	2	23	24	25	21	22
7	5	3	1	4	2	40	38	36	39	37
4	1	2	3	4	5	21	22	23	24	25
3	3	4	5	1	2	18	19	20	16	17
6	5	3	1	4	2	35	33	31	34	32
3	3	4	5	1	2	18	19	20	16	17
1	1	2	3	4	5	6	7	8	9	10
9	5	3	1	4	2	50	48	46	49	47

Λ	κ					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
5	3	4	5	1	2	28	29	30	26	27
8	5	3	1	4	2	45	43	41	44	42

Table 4. Tables for $C_4 \cup P_2 \to C_{20} \cup 5P_2$

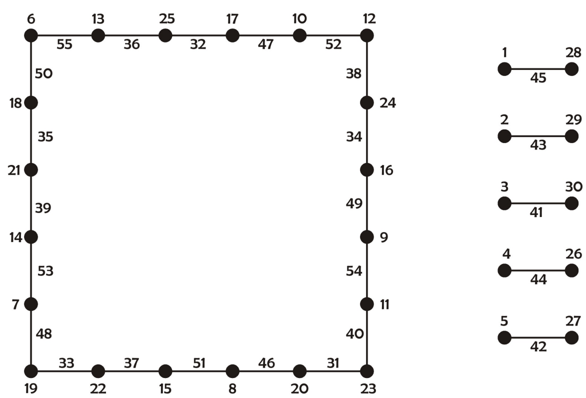

Figure 9. SEMT labeling for $C_{20} \cup 5P_2$

Applying Theorem 3 to an SEMT labeling Cartesian product of $P_2 \square P_n$ (*ladder* graph) that is given in [12] gives an SEMT labeling of union of odd number of ladders $m\,(P_2 \square P_n)$ for any odd values of m. However, ladder graphs are 3-colorable so the preservation of its SEMT labeling is already guaranteed by Theorem 2.

In [11], application of Theorem 3 to other families such as fans, wheels, umbrellas, tadpoles, braids, and many other families of graphs are given. One of the interesting possibilities of this work is to find a way to apply Theorem 3 using a certain combination of κ, κ', and probably other variations of Kotzig arrays, to solve the open problem of finding the EMT or SEMT labeling of unions of odd number of wheels.

In [2], Enomoto et al. checked all wheels up tp $n = 29$ and found that W_n has EMT labeling if $n \not\equiv 3 \bmod 4$. The construction of EMT labeling of W_n for all other cases are given in [7, 10]. When n is even wheels W_n are 3-colorable, so the existence of EMT labeling of tW_n for odd t and even n is guaranteed by Theorem 2. Also, it is given in [10] that tW_n does not have EMT labeling when t is odd and $n \equiv 3 \bmod 4$.

3. Conclusion

We conclude this paper with the following open problems.

Problem 1. *Find an EMT labeling for general* $m\left(C_n \circ \overline{K_2}\right)$ *when* $m \geq 0$.

Problem 2. *Find an EMT labeling for* tW_n *when* t *is odd and* $n \equiv 1 \bmod 4$*, or prove that it cannot be done.*

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