On local antimagic vertex coloring of corona products related to friendship and fan graph

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Abstract
Let $G = (V, E)$ be connected graph. A bijection $f : E \rightarrow \{1, 2, 3, \ldots, |E|\}$ is a local antimagic of $G$ if any adjacent vertices $u, v \in V$ satisfies $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$, $E(u)$ is the set of edges incident to $u$. When vertex $u$ is assigned the color $w(u)$, we called it a local antimagic vertex coloring of $G$. A local antimagic chromatic number of $G$, denoted by $\chi_{la}(G)$, is the minimum number of colors taken over all colorings induced by the local antimagic labeling of $G$. In this paper, we determine the local antimagic chromatic number of corona product of friendship and fan with null graph on $m$ vertices, namely, $\chi_{la}(F_n \odot K_m)$ and $\chi_{la}(f(1,n) \odot K_m)$.

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1. Introduction
All graphs $G = (V, E)$ considered in this paper are simple and finite. A vertex coloring of a graph $G$ is an assignment of color to vertices of $G$ such that every two adjacent vertices have a different color. A $k$-coloring of $G$ is defined as a map $h : V \rightarrow \{1, 2, \ldots, k\}$ such that $h(u) \neq h(v)$ for any adjacent vertices $u, v \in V$. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest positive integer $k$ assigned to $G$.

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Hartsfield and Ringel [6] introduced the principle of antimagic labeling, and then Gallian [4] surveyed the researches conducted on graph labeling and its variation, including antimagic labeling. The antimagic on a graph is defined as follows. Let \( f : E \to \{1, 2, 3, \ldots, |E|\} \) be a bijection. The weight of vertex \( u \), denoted \( w(u) \), is defined as \( w(u) = \sum_{e \in E(u)} f(e) \), where \( E(u) \) is the set of edges incident to \( u \). The graph \( G \) is called antimagic if \( w(u) \neq w(v) \), for every two vertices \( u, v \in V \). Arumugam et al. [1] introduced the term of local antimagic as follows. A graph \( G \) is called local antimagic if \( w(u) \neq w(v) \), for any adjacent vertices \( u, v \in V \). If for every distinct weight we assign distinct color, then it is called local antimagic vertex coloring. The local chromatic number of \( G \), denoted by \( \chi_{la}(G) \), is the minimum number of colors taken over all colorings induced by local antimagic labeling of \( G \). The local chromatic number of some graphs has been discovered, such as a tree, path, cycle, friendship, complete bipartite, an amalgamation of paths, wheel [3], kite, and cycle with two pendants [9].

Putri et al. [11] initiated a variation of local antimagic coloring named local antimagic total vertex labeling where the label is assigned to the vertices and edges of \( G \). The weight of vertex \( u \in V \), \( w(u) \), is the sum of labels of all edges incident with \( u \) and the label of \( u \) itself. A local antimagic total chromatic number of \( G \), denoted by \( \chi_{lat}(G) \), is the minimum number of colors induced by local vertex antimagic total labeling of \( G \). The local antimagic total chromatic number of some graphs have been discovered such as star, a double star, banana tree, centipede, amalgamation of graphs [11] and the corona product of some graphs with \( K_2 \) [7].

If the vertices of \( G \) received the smaller label in the local antimagic total labeling, then it is called the super local antimagic total. While, when the smaller labels are assigned to edges of \( G \), it is called super edge local antimagic total labeling. The super local antimagic chromatic number and the super edge local antimagic chromatic number of \( G \) is denoted by \( \chi_{slat}(G) \) and \( \chi_{selat}(G) \) respectively. The super local antimagic chromatic number of some graphs have been discovered for path and its derivation, hedge, hedgerow, star, and an amalgamation of graphs [5].

A corona product of \( H \) and \( G \), denoted by \( G \odot H \), is a graph obtained by taking one copy of \( G \) along with \( |V(G)| \) copies of \( H \) and putting extra edges making the \( i \)-th vertex of \( G \) adjacent to every vertex of the \( i \)-th copy of \( H \) [3]. A null graph on \( m \) vertices, denoted by \( K_m \), as a graph that has \( m \) isolated vertices [2].

In this paper, we study the local antimagic chromatic number of corona products of friendship and fan with a null graph on \( m \) vertices. Arumugam et al. [1] proved a sharp lower bound for any tree, and Lau et al. [8] generalized the theorem as follows.

**Theorem 1.1.** [8] Let \( G \) be a graph having \( k \) pendants. If \( G \) is not \( K_2 \), the \( \chi_{la}(G) \geq k + 1 \) and the bound is sharp.

2. Main Results

2.1. Corona Products of Friendship and Null Graphs

A friendship graph \( F_n \) can be constructed by joining \( n \) copies of \( C_3 \) with a common vertex. Figure 1 illustrates the graph \( F_n \odot K_m \). Since \( F_1 \cong C_3 \) and Arumugam et al. [2] already give
Theorem 2.1. Let $F_n$ be fan graph on $n$ cycles and $K_m$ null graph on $m$ vertices. For $n \geq 2$ and $m \geq 1$, $\chi_{la}(F_n \odot K_m) = m(2n + 1) + 3$.

Proof. Let $V(F_n \odot K_m) = \{x, v_i, u_i, v^i_j, x^i_j | 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ and $E(F_n \odot K_m) = \{xu_i, xv_i, xx_j, u_iv_i, u_iu^i_j, v^i_jv^i_j | 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$.

For the upper bound, we show that $\chi_{la}(F_n \odot K_m) \leq m(2n + 1) + 3$. Define $f : E \to \{1, 2, \ldots, m(2n + 1) + 3n\}$. Label $u_iv_i, xu_i, xv_i$, and $xx_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

\[
\begin{align*}
    f(u_iv_i) &= i, \\
    f(xu_i) &= 3n + 2 - 2i, \\
    f(xv_i) &= 3n + 1 - 2i, \\
    f(xx_j) &= 2mn + 3n + j.
\end{align*}
\]

Then, to label $u_iu^i_j$ and $v^i_jv^i_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, we divide into two cases according to parity of $m$.

Case 1. $m$ is odd
For $1 \leq i \leq n$ and $1 \leq j \leq m$, label the edges as follows.

$$f(u_iu_j) = \begin{cases} 
  n(2j + 1) + i, & j \text{ is odd}, \\
  2n(j + 1) + 1 - i, & j \text{ is even},
\end{cases}$$

$$f(v_iv_j) = \begin{cases} 
  2n(j + 1) + i, & j \text{ is odd}, \\
  n(2j + 3) + 1 - i, & j \text{ is even}.
\end{cases}$$

The labeling $f$ is obviously a local antimagic with weights as follows.

$$w(u_i) = m^2n + mn + 4n + 2 + \frac{3mn + m - 3n - 1}{2},$$

$$w(v_i) = m^2n + 2mn + 2n + 1 + \frac{3mn + m + n - 1}{2},$$

$$w(x) = 4n^2 + n + 2m^2n + 3mn + \frac{m^2 + m}{2},$$

$$w(x_j) = 2mn + 3n + j,$$

$$w(u_j^i) = \begin{cases} 
  n(2j + 1) + i, & j \text{ is odd}, \\
  2n(j + 1) + 1 - i, & j \text{ is even},
\end{cases}$$

$$w(v_j^i) = \begin{cases} 
  2n(j + 1) + i, & j \text{ is odd}, \\
  n(2j + 3) + 1 - i, & j \text{ is even}.
\end{cases}$$

**Case 2.** $m$ is even

For $1 \leq i \leq n$ and $1 \leq j \leq m$, label the edges as follows.

$$f(u_iu_j) = \begin{cases} 
  3n + 2i - 1, & j = 1, \\
  n(2j + 1) + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
  2n(j + 1) + 1 - i, & j \text{ is even},
\end{cases}$$

$$f(v_iv_j) = \begin{cases} 
  3n + 2i, & j = 1, \\
  2n(j + 1) + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
  n(2j + 3) + 1 - i, & j \text{ is even}.
\end{cases}$$

The labeling $f$ is obviously a local antimagic with weights as follows.

$$w(u_i) = m^2n + 2mn + 3n + 1 + \frac{mn + m}{2},$$

$$w(v_i) = m^2n + mn + 2n + 1 + \frac{5mn + m}{2},$$

$$w(x) = 4n^2 + n + 2m^2n + 3mn + \frac{m^2 + m}{2},$$

$$w(x_j) = 2mn + 3n + j,$$

$$w(u_j^i) = \begin{cases} 
  3n + 2i - 1, & j = 1, \\
  n(2j + 1) + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
  2n(j + 1) + 1 - i, & j \text{ is even},
\end{cases}$$

$$w(v_j^i) = \begin{cases} 
  3n + 2i, & j = 1, \\
  2n(j + 1) + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
  n(2j + 3) + 1 - i, & j \text{ is even}.
\end{cases}$$

Note that for $1 \leq i \leq n, 1 \leq j \leq m$, the weights of $u_j^i$, $v_j^i$, and $x_j$ depend on $i$ and $j$ while the weight of $u_i, v_i,$ and $x$ are constant. Hence, we have $2mn + m + 3$ different weights in total.
Therefore, $\chi_{la}(F_n \odot \overline{K_m}) \leq m(2n + 1) + 3$.

For the lower bound, we show that $\chi_{la}(F_n \odot \overline{K_m}) \geq m(2n + 1) + 3$. Since $F_n \odot \overline{K_m}$ has $2mn + m$ pendants, by using Theorem 1.1, we have $\chi_{la}(F_n \odot \overline{K_m}) \geq 2mn + m + 1$. Suppose $\chi_{la}(F_n \odot \overline{K_m}) \geq 2mn + m + 1$. Then, $w(x)$ will equal to either $w(u_j^i)$ or $w(v_j^i)$ for some $i$ and $j$.

Since $d(x) = 2n + m$, we obtain $w(x) \geq \sum_{k=1}^{2n+m} k = \frac{(2n+m)(2n+m+1)}{2}$, where $d(x)$ is degree of vertex $x$. On the other hand, the weights of either $w(u_j^i)$ or $w(v_j^i)$ is bounded by $2mn + m + 3n$ which implies $w(x) \geq \frac{(2n+m)(2n+m+1)}{2} = 2n^2 + 2mn + n + \frac{m^2 + m}{2} \geq 2mn + (2n + 1)n + m > 2mn + 3n + m$. It is a contradiction. Therefore, the color of $w(x)$ must be different from all pendants and now we have extended the lower bound to $\chi_{la}(F_n \odot \overline{K_m}) \geq 2mn + m + 2$.

Suppose $\chi_{la} \geq 2mn + m + 2$. Then, either $w(u_i) = w(v_j^i)$ or $w(v_i) = w(u_j^i)$ must be satisfy for some $j$. Suppose $w(u_i) = w(v_j^i)$. Notice that $w(u_i) \geq \frac{\sum_{k=1}^{2n+m} k}{n} = \frac{(2n+m)(2n+m+1)}{2n}$, while $w(v_j^i) \leq 2mn + m + 3n$. It is not hard to verify that $\frac{(2n+m)(2n+m+1)}{2n} = 2mn + (\frac{m+1}{2})m + 4n + 2 > 2mn + m + 3n$ for $n \geq 2$ and $m \geq 1$. It is a contradiction since $w(v_j^i) < w(u_i)$. We can construct the same argument to show a contradiction for the case $w(u_i) = w(x_j)$ or $w(v_i) = w(x_j)$ for some $j$. Therefore, the color of $w(v_i)$ must be different from all pendants and now we have extended the lower bound to $\chi_{la}(F_n \odot \overline{K_m}) \geq 2mn + m + 3$.

Since both inequalities $\chi_{la}(F_n \odot \overline{K_m}) \leq 2mn + m + 3$ and $\chi_{la}(F_n \odot \overline{K_m}) \geq 2mn + m + 3$ hold, then $\chi_{la}(F_n \odot \overline{K_m}) = 2mn + m + 3$.

We give the local antimagic vertex coloring for $F_5 \odot \overline{K_3}$ with $\chi_{la}(F_5 \odot \overline{K_3}) = 36$ in Figure 2.

2.2. Corona Products of Fan and Null Graphs

A fan graph $f_{(1,n)}$ is defined as the graph $K_1 + P_n$ where $K_1$ is the null graph on one vertex and $P_n$ is the path graph on $n$ vertices. Figure 3 illustrates the graph $f_{(1,n)} \odot \overline{K_m}$. Since $f_{(1,2)} \cong C_3$ and Arumugam et al. [2] already give $\chi_{la}(C_n \odot \overline{K_m})$, here we consider $f_{(1,n)} \odot \overline{K_m}$ for $n \geq 3$ and $m \geq 1$.

Theorem 2.2. Let $f_{(1,n)}$ be friendship of $n + 1$ vertices and $\overline{K_m}$ null graph on $m$ vertices. For $n \geq 3$ and $m \geq 1$, $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) = m(n + 1) + 3$.

Proof. Let $V(f_{(1,n)} \odot \overline{K_m}) = \{x, v_i, v_j^i, x_j | 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ and $E(f_{(1,n)} \odot \overline{K_m}) = \{xx_j, xv_i, v_i v_{i+1}, v_i v_j^i | 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$.

For the upper bound, we show that $\chi_{la}(f_{(1,n)} \odot \overline{K_m}) \leq m(n + 1) + 3$. Define $f : E \rightarrow \{1, 2, \ldots, m(n + 1) + 2n - 1\}$. We divide into two cases depend on the parity of $n$.

Case 1. $n$ is odd
Figure 2. The local antimagic vertex coloring of $F_5 \odot K_3$

Figure 3. The graph $f_{(1,n)} \odot \overline{K_m}$
Label the edges $v_i v_{i+1}, xv_i,$ and $x v_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$f(v_i v_{i+1}) = \begin{cases} \frac{i+1}{2}, & i \text{ is odd}, \\ n - 1 - \frac{i-2}{2}, & i \text{ is even}, \end{cases}$$

$$f(xv_i) = \begin{cases} 2n - 1, & i = 1, \\ n + \frac{i-3}{2}, & i \neq 1 \text{ and } i \text{ is odd}, \\ n - 1 + \frac{n-1+i}{2}, & i \text{ is even}, \end{cases}$$

$$f(x x_j) = mn + m + 2n - j.$$

Then, to label $v_i v_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, we divide into two subcases according to parity of $m$.

**Subcase 1. $m$ is odd**
Label $v_i v_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$f(v_i v_j^i) = \begin{cases} 2n + 1 + \frac{n-3}{2}, & i = n, \\ 2n + \frac{n-i-2}{2}, & i \neq n \text{ and } i \text{ is odd}, \\ 3n - 1 - \frac{i-2}{2}, & i \text{ is even}, \end{cases}$$

$$f(v_i v_j^i) = \begin{cases} (j+1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\ (j+2)n - i, & j \text{ is even}. \end{cases}$$

The labeling $f$ is obviously a local antimagic with weights

$$w(v_i) = \begin{cases} 2mn + 2n - 1 + \frac{m^2-m}{2}, & i \text{ is odd}, \\ 2mn + 3n - 1 + \frac{m^2-m}{2}, & i \text{ is even}. \end{cases}$$

$$w(x) = m^2n + 2mn + \frac{m^2-m+3n^2-n}{2},$$

$$w(x_j) = mn + m + 2n - j,$$

$$w(v_j^i) = \begin{cases} 2n + 1 + \frac{n-3}{2}, & j = 1; i = n, \\ 2n + \frac{n-i-2}{2}, & j = 1; i \neq n \text{ and } i \text{ is odd}, \\ 3n - 1 - \frac{i-2}{2}, & j = 1; i \text{ is even}, \\ (j+1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\ (j+2)n - i, & j \text{ is even}. \end{cases}$$

**Subcase 2. $m$ is even**
Label $v_i v_j^i$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

$$f(v_i v_j^i) = \begin{cases} 3n - 1, & i = n, \\ 3n - 2 - i, & i \neq n \text{ and } i \text{ is odd}, \\ 3n - i, & i \text{ is even}, \end{cases}$$

$$f(v_i v_j^i) = \begin{cases} 3n, & i = n, \\ 3n + \frac{i+1}{2}, & i \neq n \text{ and } i \text{ is odd}, \\ 3n + \frac{n-1+i}{2}, & i \text{ is even}, \end{cases}$$

$$f(v_i v_j^i) = \begin{cases} (j+1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\ (j+2)n - i, & j \neq 2 \text{ and is even}. \end{cases}$$

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The labeling $f$ is obviously a local antimagic with weights

\[
\begin{align*}
  w(v_i) &= \begin{cases} 
    2mn + 2n + 1 + \frac{m^2n-m}{2}, & i \text{ is odd}, \\
    2mn + 3n - 1 + \frac{m^2n-m}{2}, & i \text{ is even},
  \end{cases} \\
  w(x) &= m^2n + 2mn + \frac{m^2-m+3n^2-n}{2}, \\
  w(x_j) &= mn + m + 2n - j,
\end{align*}
\]

\[
\begin{align*}
  w(v^i_j) &= \begin{cases} 
    3n - 1, & j = 1; i = n, \\
    3n - 2 - i, & j = 1; i \neq n \text{ and } i \text{ is odd}, \\
    3n - i, & j = 1; i \text{ is even}, \\
    3n, & j = 2; i = n, \\
    3n + \frac{i+1}{2}, & j = 2; i \neq n \text{ and } i \text{ is odd}, \\
    3n + \frac{n-1+i}{2}, & j = 2; i \text{ is even}, \\
    (j + 1)n + 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
    (j + 2)n - i, & j \neq 2 \text{ and is even}.
  \end{cases}
\end{align*}
\]

**Case 2.** $n$ is even
Label the edges $\{v_iv_{i+1}, xv_i, \text{ and } xx_j\}$ $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

\[
\begin{align*}
  f(v_i v_{i+1}) &= \begin{cases} 
    \frac{i+1}{2}, & i \text{ is odd}, \\
    n - 1 - \frac{i-2}{2}, & i \text{ is even}, \\
    2n - 1, & i = 1,
  \end{cases} \\
  f(xv_i) &= \begin{cases} 
    2n - 2, & i = n, \\
    n + \frac{i+3}{2}, & i \neq 1 \text{ and } i \text{ is odd}, \\
    n - 2 + \frac{n+i}{2}, & i \neq n \text{ and } i \text{ is even},
  \end{cases} \\
  f(xx_j) &= mn + m + 2n - j.
\end{align*}
\]

Then, to label $v_iv^i_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, we divide into two subcases according to parity of $m$.

**Subcase 1.** $m$ is odd
Label $v_iv^i_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows.

\[
\begin{align*}
  f(v_i v^i_j) &= \begin{cases} 
    3n - 1, & i = n, \\
    3n - 1 - \frac{i}{2}, & i \neq n \text{ and } i \text{ is even}, \\
    2n + \frac{n-i-1}{2}, & i \text{ is odd},
  \end{cases} \\
  f(v^i_j) &= \begin{cases} 
    (j + 1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
    (j + 2)n - i, & j \text{ is even}.
  \end{cases}
\end{align*}
\]
The labeling \( f \) is obviously a local antimagic with weights
\[
\begin{align*}
w(v_i) &= \begin{cases} 
2mn + 2n - 1 + \frac{m^2n-m+1}{2}, & i \text{ is odd}, \\
2mn + 3n - 3 + \frac{m^2n-m+1}{2}, & i \text{ is even},
\end{cases} \\
w(x) &= m^2n + 2mn + \frac{m^2-m+3n^2-n}{2}, \\
w(x_j) &= mn + m + 2n - j,
\end{align*}
\]
\[
\begin{align*}
w(v_j^i) &= \begin{cases} 
3n - 1, & j = 1; i = n, \\
3n - 1 - i, & j = 1; i \neq n \\
3n + 1 + \frac{n-2}{2}, & i = n, \\
3n + 1 + \frac{n+i-2}{2}, & i \neq n \text{ and } i \text{ is even}, \\
3n + \frac{i-1}{2}, & i \text{ is odd}, \\
(j + 1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
(j + 2)n - i, & j \text{ is even}.
\end{cases}
\end{align*}
\]

**Subcase 2.** \( m \) is even
Label \( v_i v_j^i \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) as follows
\[
\begin{align*}
f(v_i v_j^i) &= \begin{cases} 
3n - 1, & i = n, \\
3n - 1 - i, & i \neq n,
\end{cases} \\
f(v_i v_j^2) &= \begin{cases} 
3n + 1 + \frac{n-2}{2}, & i = n, \\
3n + 1 + \frac{ni-2}{2}, & i \neq n \text{ and } i \text{ is even}, \\
3n + \frac{i-1}{2}, & i \text{ is odd},
\end{cases} \\
f(v_i v_j^j) &= \begin{cases} 
(j + 1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
(j + 2)n - i, & j \neq 2 \text{ and } j \text{ is even}.
\end{cases}
\end{align*}
\]

The labeling \( f \) is obviously a local antimagic with weights
\[
\begin{align*}
w(v_i) &= \begin{cases} 
2mn + 2n - 1 + \frac{m^2n-m}{2}, & i \text{ is odd}, \\
2mn + 3n - 2 + \frac{m^2n-m}{2}, & i \text{ is even},
\end{cases} \\
w(x) &= m^2n + 2mn + \frac{m^2-m+3n^2-n}{2}, \\
w(x_j) &= mn + m + 2n - j,
\end{align*}
\]
\[
\begin{align*}
w(v_j^i) &= \begin{cases} 
3n - 1, & j = 1; i = n, \\
3n - 1 - i, & j = 1; i \neq n, \\
3n + 1 + \frac{n-2}{2}, & j = 2; i = n, \\
3n + 1 + \frac{n+i-2}{2}, & j = 2; i \neq n \text{ and } i \text{ is even}, \\
3n + \frac{i-1}{2}, & j = 2; i \text{ is odd},
\end{cases} \\
& \begin{cases} 
(j + 1)n - 1 + i, & j \neq 1 \text{ and } j \text{ is odd}, \\
(j + 2)n - i, & j \neq 2 \text{ and } j \text{ is even}.
\end{cases}
\end{align*}
\]

Since for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), the weights of \( v_j^i \) and \( xx_j \) depend on \( i \) and \( j \), while the weight of \( v_i \) and \( x \) are constant, we have \( mn + m + 3 \) different weights in total. Therefore, \( \chi_{la}(f(1,n) \circ K_m) \leq mn + m + 3 \).

For the lower bound, we show that \( \chi_{la}(f(1,n) \circ K_m) \geq mn + m + 3 \). Since \( f(1,n) \circ K_m \) has \( mn + m \) pendants, by using Theorem 1.1, we have \( \chi_{la}(f(1,n) \circ K_m) \geq mn + m + 1 \). Suppose
\( \chi_{la}(f(1,n) \odot \overline{K}_m) \geq mn + m + 1 \). Then, \( w(x) \) must equal to \( w(v^i_j) \) for some \( i \) and \( j \). Note that \( w(x) \geq \sum_{k=1}^{m+n} k = \frac{(m+n)(m+n+1)}{2} \), while \( w(v^i_j) \leq mn + m + 2n - 1 \) for any \( i \) and \( j \). It is not hard to verify that \( \frac{(m+n)(m+n+1)}{2} = mn + (\frac{m+n}{2})n > mn + m + 2n - 1 \), if \( n \geq 3 \). Hence, we get a contradiction. Therefore, \( \chi_{la}(f(1,n) \odot \overline{K}_m) \geq mn + m + 2 \).

Now, suppose \( \chi_{la}(f(1,n) \odot \overline{K}_m) \geq mn + m + 2 \). Since \( w(x) \) is unique, there must be at least \( \lceil \frac{n}{\chi} \rceil \) pairs of vertices such that \( w(v_i) = w(v^i_j) \) for some \( i \) and \( j \). We will show that it is impossible by considering the parity of \( n \). First, if \( n \) is even, \( w(v^i_j) \geq \frac{\sum_{k=1}^{m+n+2} k}{2} = \frac{(m+n+3n+2)(m+n+4)}{2} \) for all \( i \), while \( w(v^i_j) \leq (n+1)m + 2n - 1 \) for all \( i \) and \( j \). It is not hard to verify that \( \frac{(3n+3)2}{2} + \frac{(9n+9+1)2}{2} \) for all \( i \), while \( w(v^i_j) \leq (n+1)m + 2n - 1 \) for all \( i \) and \( j \). It is not hard to verify that \( \frac{(3n+3)m + (\frac{m^2+9}{4})n + (\frac{9}{2} + \frac{n}{n})}{2} \) for all \( i \), while \( w(v^i_j) \leq (n+1)m + 2n - 1 \) for all \( i \) and \( j \). It is not hard to verify that \( \frac{(3n+3)2}{2} + \frac{(9n+9+1)2}{2} \) for all \( i \), while \( w(v^i_j) \leq (n+1)m + 2n - 1 \) for all \( i \) and \( j \). Therefore, \( \chi_{la}(f(1,n) \odot \overline{K}_m) \geq mn + m + 3 \).

Since both \( \chi_{la}(f(1,n) \odot \overline{K}_m) \leq mn + m + 3 \) and \( \chi_{la}(f(1,n) \odot \overline{K}_m) \geq mn + m + 3 \) hold, then \( \chi_{la}(f(1,n) \odot \overline{K}_m) = mn + m + 3 \).

We give the local antimagic vertex coloring for \( f(1,6) \odot \overline{K}_3 \) with \( \chi_{la}(f(1,6) \odot \overline{K}_3) = 24 \) in Figure 4.

![Figure 4. The local antimagic vertex coloring of \( f(1,6) \odot \overline{K}_3 \)](image-url)
3. Conclusion

We summarise the results in Table 1.

<table>
<thead>
<tr>
<th>Corona Products of</th>
<th>Notation</th>
<th>$\chi_{la}$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friendship with Null graph on $m$ vertices</td>
<td>$F_n \odot \overline{K}_m$</td>
<td>$m(2n+1) + 3$</td>
<td>$n \geq 2$ and $m \geq 1$</td>
</tr>
<tr>
<td>Fan with Null graph on $m$ vertices</td>
<td>$f_{(1,n)} \odot \overline{K}_m$</td>
<td>$m(n+1) + 3$</td>
<td>$n \geq 3$ and $m \geq 1$</td>
</tr>
</tbody>
</table>

References


