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Another H-super magic decompositions of the lexicographic product of graphs

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Abstract

Let H and G be two simple graphs. The concept of an H-magic decomposition of G arises from the combination between graph decomposition and graph labeling. A decomposition of a graph G into isomorphic copies of a graph H is H-magic if there is a bijection $f : V(G) \cup E(G) \longrightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that the sum of labels of edges and vertices of each copy of H in the decomposition is constant. A lexicographic product of two graphs G_1 and G_2, denoted by $G_1[G_2]$, is a graph which arises from G_1 by replacing each vertex of G_1 by a copy of the G_2 and each edge of G_1 by all edges of the complete bipartite graph $K_{n,n}$ where n is the order of G_2. In this paper we provide a sufficient condition for $\overline{C_n}[\overline{K_m}]$ in order to have a $P_t[\overline{K_m}]$-magic decompositions, where $n > 3, m > 1$, and $t = 3, 4, n - 2$.

Keywords: complement of graph, lexicographic product, H-magic decomposition
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1. Introduction

Let G be a simple graph and H be a subgraph of G. A decomposition of G into isomorphic copies of H is called $H-$ magic if there is a bijection $f : V(G) \cup E(G) \longrightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that the sum of labels of edges and vertices of each copy of H in the decomposition is

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constant. A lexicographic product of two graphs G_1 and G_2 is defined as graph which constructed from the graph G_1 and then replacing each vertex of G_1 by a copy of G_2 and each edge of G_1 by edges of complete bipartite graph $K_{n,n}$, where $|V(G)| = n$. The lexicographic product of G_1 and G_2 with this construction is denoted by $G_1[G_2]$ [1].

A labeling of a graph $G = (V, E)$ is a bijection from a set of elements of graphs to a set of numbers. The edge magic and super edge magic labelings were first introduced by Kotzig and Roza [9] and Enomoto, Lladò, Nakamigawa, and Ringel [3], respectively. There are some results in edge magic and super edge magic, such as in [2, 3, 12, 13]. The notion of an $H-$ (super) magic labeling was introduced by Gutièrrez and Lladò [5] in 2005. In 2010, Maryati and Salman [11] used multiset partition concept to obtain a super magic labeling of path amalgamation of isomorphic graphs. Inayah et al. [8]have improved the concept of labeling graphs became $H-$(anti) magic decomposition. In almost the same time, Liang [10] discused cycle-supermagic decompositions of complete multipartite graphs and in 2015, Hendy [6] has discused the $H-$ super(anti)magic decompositions of antiprism graphs. For a complete results in graph labeling, see [4].

In this research we interest in decomposing the lexicographic product of graphs $\overline{C_n}[\overline{K_m}]$ then labeling of the edges and vertices of each isomorphic copies of $P_t[\overline{K_m}]$ to obtain $P_t[\overline{K_m}]-$ magic decomposition, where $n > 3, m > 1$, and $t = 3, 4, n-2$.

Preliminaries

Let G be a simple graph. Complement of G, denoted by $\overline{G}$, is graph which $V(\overline{G}) = V(G)$ and $\forall u, v \in V(G)$ uv is edge of $\overline{G}$ if and only if uv is not edge of G. A family $\mathbb{B} = \{G_1, G_2, ..., G_t\}$ of subgraphs of G is an H-decomposition of G if all subgraphs are isomorphic to graph H, $E(G_i) \cap E(G_j) = \emptyset$, for $i \neq j$, and $\bigcup_{i=1}^{t} E(G_i) = E(G)$. In such case, we write $G = G_1 \oplus G_2 \oplus ... \oplus G_t$ and G is said to be H-decomposable. if G is an H-decomposable graph, then we also write $H|G$.

Let $\mathbb{B}$ is an H-decomposition of G. The graph G is said to be H-magic if there exists a bijection $f : V(G) \cup E(G) \longrightarrow \{1, 2, ..., |V(G) \cup E(G)|\}$ such that $\forall B \in \mathbb{B}$, $\sum_{v \in V(B)} f(v) + \sum_{e \in E(B)} f(e)$ is constant. Such a function f is called an H-magic labeling of G. The sum of all the vertex and edges labels of H (under a labeling f) is denoted by $\sum f(H)$. The constant value that every copy of H takes under the labeling f is denoted by $m(f)$.

The one of the concept of multi set partition, k-balance multi set, was presented by Maryati et al. [11]. In this paper, $\sum_{x \in X} x$, denoted by $\sum X$. Multi set is a set which may has the same elements. For positive integer n and k_i with $i \in [1, n]$, multi set $\{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}\}$ is a set which has k_i elements a_i for $i \in [1, n]$. Suppose V and W are two multi sets with $V = \{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}\}$ and $W = \{b_1^{l_1}, b_2^{l_2}, ..., b_m^{l_m}\}$. Defined by: $V \biguplus W = \{a_1^{k_1}, a_2^{k_2}, ..., a_n^{k_n}, b_1^{l_1}, b_2^{l_2}, ..., b_m^{l_m}\}$. Let $k \in N$ and Y is a multi set of positive integers. Y is a k-balance multi set if there exists k subsets of Y such as: $Y_1, Y_2, ..., Y_k$, such that for all $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\sum Y_i = \frac{\sum Y}{k} \in N$ and $\biguplus_{i=1}^{k} Y_i = Y$.

Lemma 1.1. *[7] $P_n[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ if and only if $P_n|\overline{C_n}$*

Lemma 1.2. *[7] Let t be any integer with $t > 1$. If $P_t[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ then $n(n-3) \equiv 0(mod 2(t-1))$*

Theorem 1.1. *[7] Let n and m be integers with $n > 3$ and $m > 1$. The graph $\overline{C_n}[\overline{K_m}]$ has $P_2[\overline{K_m}]$-super magic decomposition if and only if m is even or m is odd and $n \equiv 1(mod 4)$, or m is odd and $n \equiv 2(mod 4)$, or m is odd and $n \equiv 3(mod 4)$.*

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2. Results

Lemma 2.1. $P_3[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ *if and only if* $n \neq 4$, $n \equiv 0(mod4)$ *or* $n \equiv 3(mod4)$.

Proof. $(\Rightarrow)$ Let $P_3[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$, then from Lemma 2.1 we have that $P_3|\overline{C_n}$. From Lemma 2.2 we have that $n \equiv 0(mod4)$ or $n \equiv 3(mod4)$. Because of $\overline{C_4}$ doesn't have P_3, this is not occur for $n = 4$.
$(\Leftarrow)$ Now let $n \neq 4$, $n \equiv 0(mod4)$ dan $V(\overline{C_n}) = \{v_1, ..., v_{4k}\}$, $k \in Z^+$. Let $N(v_i)$=$V(\overline{C_n}) \setminus \{v_{i-1}, v_{i+1}\}$. Follow this algorithm decompose $\overline{C_n}$.

Algorithm 1:

1 Choose the path $P_1 : v_3 - v_1 - v_4$ and let v_1 be the center of the rotation. Rotate P_1 such that v_1 on v_3, v_3 on v_5 and v_4 on v_6, thus we have $P_2 : v_5 - v_3 - v_6$. Do the next rotation until v_1 on v_5,..., v_{4i-1},..., v_{4k-1}. Then we have $2k$ of P_3-paths.

2 Choose the cycle $v_2 - v_4 - ... - v_{4k}$. Decompose this $2k$-cycle to k of P_3-paths.

3 Do the rotation again ($v_1 \to v_3 \to v_5 \to$...), with choosing two vertices which close with the vertices that is rotated in step 1. If this rotation is not the last rotation, do the rotation again until v_1 on position of v_{4k-1}, such that we have $2k$ of P_3-path. If this rotation is the last rotation, first do the rotation in step 1 until v_1 on position of v_{2k-1} such that we have k of P_3-path. Then rotate $P' = v_{n-2} - v_2 - v_{n-1}$ with v_2 as a center of this rotation until v_2 on position of v_{2k} and we have k P_3-path.

From the **Algorithm 1** above, we have that $P_3|\overline{C_n}$. Then from Lemma 2.1 $P_3[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ for $n \neq 4$, $n \equiv 0(mod4)$.

Let $n \equiv 3(mod4)$ dan $V(\overline{C_n}) = \{v_1, ..., v_{4k+3}\}$, $k \in Z^+$. Let $N(v_i) = V(\overline{C_n}) \setminus \{v_{i-1}, v_{i+1}\}$. Decompose $\overline{C_n}$ with the following steps.

Algorithm 2

Choose the path $Q_1 = v_3 - v_1 - v_4$ with v_1 is the center of rotation. Rotate Q_1 such that v_1 on v_2 and we have $Q_2 = v_4 - v_2 - v_5$. Do the next rotation such that v_1 on v_3,v_4, v_i,..., v_{4k+3}. Do the rotation such that we have kn P_3-path.
From **Algorithm 2**, it's clearly that $P_3|\overline{C_n}$. Thus from Lemma 2.1 $P_3[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ for $n \equiv 3(mod4)$. □

See Figure 1 to see graph $\overline{C_8}$ can be decomposed into 10 P_3-path.

Theorem 2.1. *Suppose* $n, m \in Z^+$ *and* $m > 1$. *For* $n \equiv 3(mod4)$, *or (*$n \equiv 0(mod4)$ *and* m *is even, Graph* $\overline{C_n}[\overline{K_m}]$ *have* $P_3[\overline{K_m}]$*-magic decomposition.*

Proof. Let $n \equiv 3(mod4)$. From Lemma 2.1 we have for $n \equiv 3(mod4)$, $P_3[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$. Let m be even. Do the next vertex labeling steps and edge labeling steps such in **case 1** in Theorem 2.1.

Let $V_1, V_2, ..., V_n$ be the partitions of $V(\overline{C_n}[\overline{K_m}])$, where $V(\overline{C_n}[\overline{K_m}]) = V_1 \cup V_2 \cup ... \cup V_n = \{v_{1,1}, v_{1,2}, ..., v_{1,m}\} \cup \{v_{2,1}, v_{2,2}, ..., v_{2,m}\} \cup ... \cup \{v_{n,1}, v_{n,2}, ..., v_{n,m}\}$. Consider the set $A^* = [1, mn] =$

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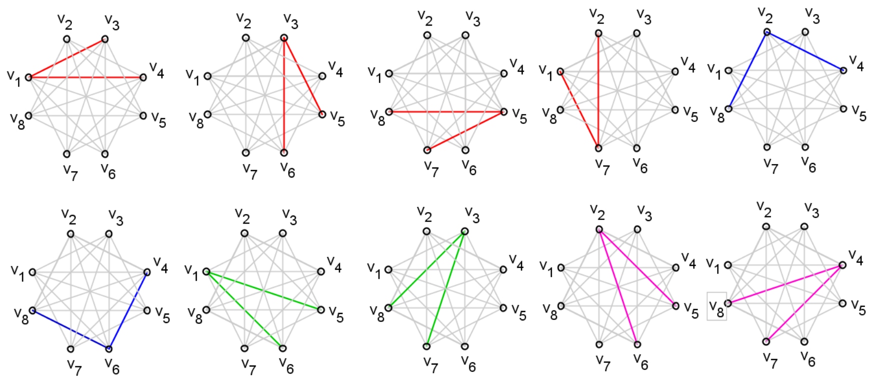

Figure 1. P_3-decomposition of $\overline{C_8}$

$[1,(2k)n], k \in Z$. for every $i \in [1,n]$, $A_i^* = \{a_j^i / 1 \le j \le m\}$, where

$$a_j^i = \begin{cases} k(j-1)+i, & \text{if j is odd;} \\ 1+nj-i, & \text{if j is even.} \end{cases}$$

is a balance subset of A^*.
Define a vertex labeling f_1 of $\overline{C_n}[\overline{K_m}]$ which will label vertices of $V_1, V_2, ..., V_n$ using elements of $A_1^*, A_2^*, ..., A_n^*$ respectively.

Consider the set $B^* = [mn+1, mn+\frac{n(n-3)m^2}{2}]$. For every $i \in [1, \frac{n(n-3}{2}]$, $B_i^* = \{b_j^i / 1 \le j \le m^2\}$, with $b_j^i = \begin{cases} mn + \frac{n(n-3)}{2}(j-1)+i, & \text{if j is odd;} \\ (mn+1)+(\frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$
$B_i^* = \{b_j^i / 1 \le j \le m^2\}$ is a balance subset of B^*. Define an edge labeling f_2 of $\overline{C_n}[\overline{K_m}]$ with label all edges in $P_2[\overline{K_m}]_i$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements in B_i^*.

Since for all $i \in [1, \frac{n(n-3)}{4}]$, $m(f_1+f_2)(P_3[\overline{K_{mi}}) = 3m(f_1) + 2m(f_2) = 3(m^2n+m) + 2(\frac{m^2}{2}(2mn+1+\frac{n(n-3)m^2}{2}) = 3m^2n+3m+m^2(2mn+1+\frac{n(n-3)m^2}{2})$ then $\overline{C_n}[\overline{K_m}]$ has $P_3[\overline{K_m}]$-magic decomposition.

Now let m is odd. Do the vertex labeling steps and edge labeling steps such in **case 4** in Theorem 2.1.

(a) Let $m=3$. Consider the set $A = [1, m(n+\frac{n(n-3)}{2})] = [1, 3(n+\frac{n(n-3)}{2})]$. For every $i \in [1, (n+\frac{n(n-3)}{2})]$, $A_i = \{a_i, b_i, c_i\}$ where

$$a_i = 1+i;$$
$$b_i = \begin{cases} (n+\frac{n(n-3)}{2}) + \lceil \frac{\frac{n(n-3)}{2})}{2} \rceil + i, & \text{for } i \in [1, \lfloor \frac{\frac{n(n-3)}{2})}{2} \rfloor]; \\ (n+\frac{n(n-3)}{2}) - \lfloor \frac{n+\frac{n(n-3)}{2}}{2} \rfloor + i, & \text{for } i \in [\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil, (n+\frac{n(n-3)}{2})]. \end{cases}$$
$$c_i = \begin{cases} 3(n+\frac{n(n-3)}{2}) + 1 - 2i, & \text{for } i \in [1, \lfloor \frac{n+\frac{n(n-3)}{2}}{2} \rfloor]; \\ 3(n+\frac{n(n-3)}{2}) + 2\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil - 2i, & \text{for } i \in [\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil, n+\frac{n(n-3)}{2}]. \end{cases}$$

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$A_i = \{a_i, b_i, c_i\}$ is a balance subset of A. Consider the set $B = [3(n + \frac{n(n-3)}{2}) + 1, 3n + (\frac{n(n-3)}{2})m^2]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $B_i = \{b^i_j / 1 \leq j \leq m^2 - 3\}$, where

$$b^i_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

$B_i = \{b^i_j / 1 \leq j \leq m^2 - 3\}$ is a balance subset of B. Define a function $h_1 : V(\overline{C_n}[\overline{K_m}]) \to \{A_i, i \in [1, n]\} \subset A$ and label all vertices in every V_i with the elements of A_i. Define a function $h_2 : E(\overline{C_n}[\overline{K_m}]) \to \{A_i, i \in [n+1, (n + \frac{n(n-3)}{2})]\} \bigcup B$ and label all edges in every $P_2[\overline{K_m}]_i$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A_{n+i} \bigcup B_i$.

(b) Let $m > 3$ and m be odd. Considering the set $A^* = [1, m(n + \frac{n(n-3)}{2})]$. Divide A^* to be two sets.

$$\begin{aligned} A &= [1, 3(n + \tfrac{n(n-3)}{2})]; \\ E &= [3(n + \tfrac{n(n-3)}{2}) + 1, m(n + \tfrac{n(n-3)}{2})]. \end{aligned}$$

Follow the same way with (a),for $m = 3$, A is a $(n + \frac{n(n-3)}{2})$-balance multi set and for every $i \in [1, (n + \frac{n(n-3)}{2})]$, A_i is a balance subset of A. Consider the set $E = [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $E_i = \{e^i_j / 1 \leq j \leq m - 3\}$, where

$$e^i_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

$E_i = \{e^i_j / 1 \leq j \leq m - 3\}$ is a balance subset of E. Considering the set $M = [m(n + \frac{n(n-3)}{2}) + 1, m^2(n + \frac{n(n-3)}{2}) + mn]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $M_i = \{m^i_j / 1 \leq j \leq m^2 - m\}$, where

$$m^i_j = \begin{cases} m(n + \frac{n(n-3)}{2}) + (\frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ m(n + \frac{n(n-3)}{2}) + 1 + (\frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

$M_i = \{m^i_j / 1 \leq j \leq m^2 - m\}$ is a balance subset of M.

Define a function $q_1 : V(\overline{C_n}[\overline{K_m}]) \to \{A^*_i = A_i \bigcup E_i, i \in [1, n]\} \subset A^*$ and label all vertices in every V_i with the elements of $\{A^*_i, i \in [1, n]\}$. Define a function $q_2 : E(\overline{C_n}[\overline{K_m}]) \to \{A^*_{n+i} = A_{n+i} \bigcup E_{n+i}\} \bigcup M$ and label all edges in every $P_2[\overline{K_m}]_i$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A^*_{n+i} \bigcup M_i$.

Since $\forall i \in [1, \frac{n(n-3)}{4}]$, $(q_1 + q_2)(P_3[\overline{K_m}]_i) = 5 \sum A^*_i + 2 \sum M_i = 5(\sum A_i + \sum E_i) = 5((2 + 4n + 2n(n-3) + \lceil \frac{2n + n(n-3)}{4} \rceil) + (\frac{m-3}{2})(3(n + \frac{n(n-3)}{2}) + 1 + m(n + \frac{n(n-3)}{2}))) + 2(\frac{m^2 - m}{2}(m(n + \frac{n(n-3)}{2}) + 1 + m^2(n + \frac{n(n-3)}{2}) + mn))$ then $\overline{C_n}[\overline{K_m}]$ has $P_3[\overline{K_m}]$-magic decomposition.

Now let $n \equiv 0(mod4)$ and m be even. From Lemma 3, we have for $n \equiv 0(mod4)$, $P_3[\overline{K_m}] | \overline{C_n}[\overline{K_m}]$. Do the vertex labeling steps and edge labeling steps such in **case 1** in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{4}]$, $m(f_1 + f_2)(P_3[\overline{K_{mi}}) = 3m(f_1) + 2m(f_2) = 3(m^2 n + m) + 2(\frac{m^2}{2}(2mn + 1 + \frac{n(n-3)m^2}{2}) = 3m^2 n + 3m + m^2(2mn + 1 + \frac{n(n-3)m^2}{2})$, then $\overline{C_n}[\overline{K_m}]$ have $P_3[\overline{K_m}]$-magic decomposition. □

Figure 2 give an example that graph $\overline{C_8}[\overline{K_2}]$ have $P_3[\overline{K_2}]$- super magic decomposition with the constant value $m(f_1 + f_2) = 503$.

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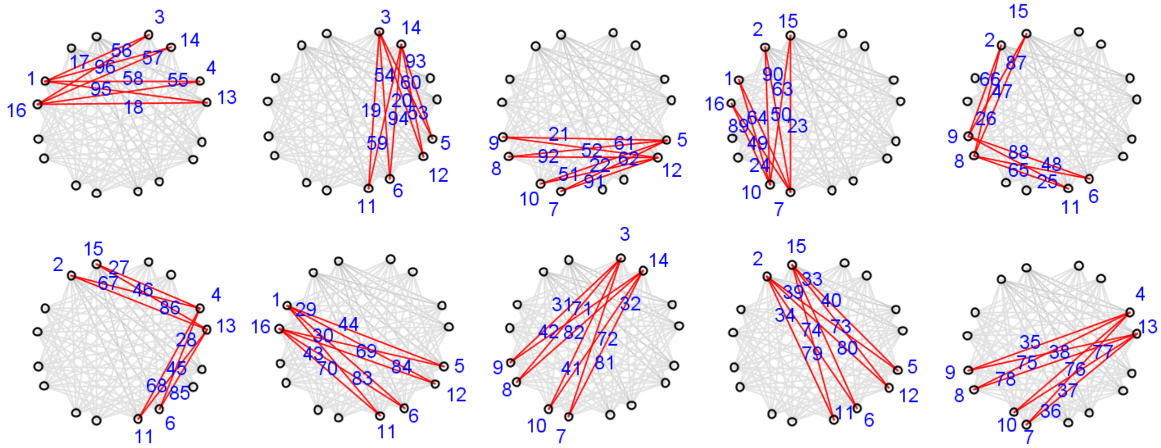

Figure 2. $P_3[\overline{K_2}]$-super magic decomposition of $\overline{C_8}[\overline{K_2}]$

Lemma 2.2. *$P_4[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$ if and only if $n \equiv 0(mod6)$ or $n \neq 3, n \equiv 3(mod6)$*

Proof. ($\Rightarrow$) Let $P_4[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$, then from Lemma 2.1, $P_4|\overline{C_n}$. From Lemma 2.2 $n \equiv 0(mod6)$ or $n \equiv 3(mod6)$. Clearly that this is not occur for $n = 3$.

($\Leftarrow$) Let $V(\overline{C_n}) = \{v_1, ..., v_{3k}\}$, $k \in Z^+$ and $N(v_i)$=$V(\overline{C_n}) \setminus \{v_{i-1}, v_{i+1}\}$. Do the algorithm 3 bellow to decompose $\overline{C_n}$.

Algorithm 3

Choose the path $R_1 : v_1 - v_3 - v_6 - v_4$ and let v_1 be the center of the rotation. Rotate R_1 such that v_1 on v_2, v_3 on v_4, v_6 on v_1 and v_4 on v_5, thus we have $R_2 = v_2 - v_4 - v_1 - v_5$. Do the next rotation such that v_1 on v_3,...etc, and redo the process until $\frac{(k-1)}{2}$ rotations. □

Figure 3 shows that graph $\overline{C_9}$ can be decompose into 9 P_4-path.

Theorem 2.2. *Let $n > 3$ and $m > 1$. For $n \equiv 3(mod12)$ or $n \equiv 6(mod12)$ or $n \equiv 9(mod12)$ or ($n \equiv 0(mod12)$ and m is even, Graph $\overline{C_n}[\overline{K_m}]$ have $P_4[\overline{K_m}]$-magic decomposition*

Proof. Let $n \equiv 3(mod12)$. From Lemma 2.2, we have that for $n \equiv 3(mod12)$, $P_4[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$. Now, let m be even. Do the next vertex labeling steps and edge labeling steps such in **case 1** in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{6}]$, $(f_1 + f_2)(P_4[\overline{K_m}_i) = 4m(f_1) + 3m(f_2) = 4(m^2n + m) + 3(\frac{m^2}{2}(2mn + 1 + \frac{n(n-3)m^2}{2})$ then $\overline{C_n}[\overline{K_m}]$ have $P_4[\overline{K_m}]$-magic decomposition.

Let m be odd. Do the next vertex labeling steps and edge labeling steps such in **case 4** in Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{6}]$, $m(q_1 + q_2)(P_4[\overline{K_m}]_i) = 7\sum A_i^* + 3\sum M_i = 7(2 + 4n + 2n(n-3) + \lceil\frac{2n+n(n-3)}{4}\rceil) + (\frac{m-3}{2})(3(n + \frac{n(n-3)}{2}) + 1 + m(n + \frac{n(n-3)}{2})) + \frac{3m^2-3m}{2}(m(n + \frac{n(n-3)}{2}) + 1 + m^2(n + \frac{n(n-3)}{2}) + mn)$, then $\overline{C_n}[\overline{K_m}]$ has $P_4[\overline{K_m}]$-magic decomposition.

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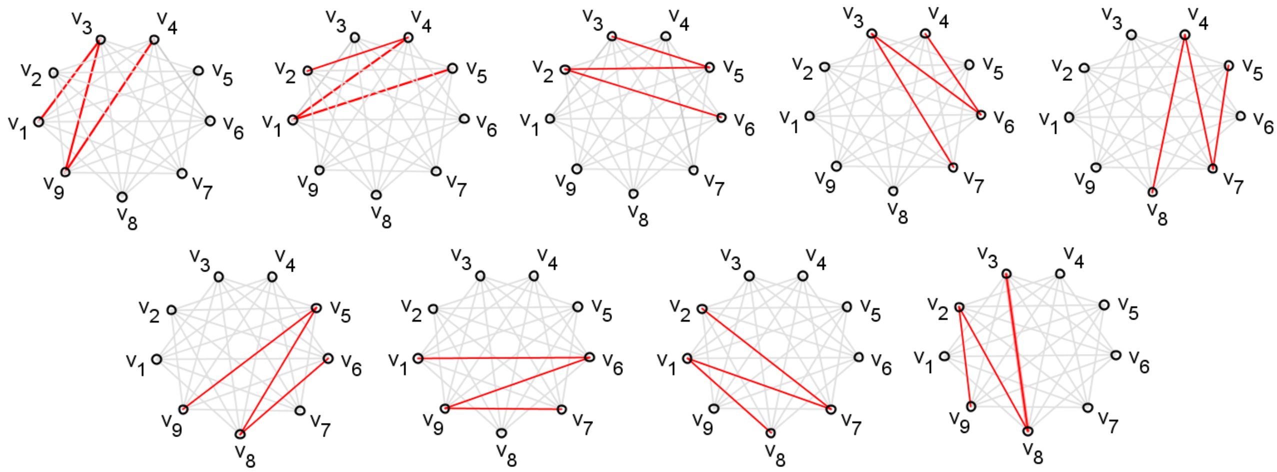

Figure 3. P_4-decomposition of $\overline{C_9}$

Let $n \equiv 6(mod12)$. From Lemma 2.2, we have that $n \equiv 6(mod12)$, $P_4[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$. Now let m is even. Do the vertex labeling steps and edge labeling steps in **case 1** Theorem 1. Because $\forall i \in [1, \frac{n(n-3)}{6}]$, $(f_1 + f_2)(P_4[\overline{K_{mi}}) = 4\sum Z_i + 3\sum X_i$ then $\overline{C_n}[\overline{K_m}]$ have $P_4[\overline{K_m}]$-magic decomposition. Let m is odd. Do the vertex labeling steps and edge labeling steps such in **case 3** in Theorem 2.1.

Let $m = 3$. Consider the set $D = [1, m(n + \frac{n(n-3)}{2})] = [1, 3(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $D_i = \{a_i, b_i, c_i\}$ where:

$$a_i = 1 + i;$$

$$b_i = \begin{cases} (n + \frac{n(n-3)}{2}) + \lceil \frac{\frac{n(n-3)}{2})}{2} \rceil + i, & \text{for } i \in [1, \lfloor \frac{\frac{n(n-3)}{2})}{2} \rfloor]; \\ (n + \frac{n(n-3)}{2}) - \lfloor \frac{n+\frac{n(n-3)}{2}}{2} \rfloor + i, & \text{for } i \in [\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil, (n + \frac{n(n-3)}{2})]. \end{cases}$$

$$c_i = \begin{cases} 3(n + \frac{n(n-3)}{2}) + 1 - 2i, & \text{for } i \in [1, \lfloor \frac{n+\frac{n(n-3)}{2}}{2} \rfloor]; \\ 3(n + \frac{n(n-3)}{2}) + 2\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil - 2i, & \text{for } i \in [\lceil \frac{n+\frac{n(n-3)}{2}}{2} \rceil, n + \frac{n(n-3)}{2}]. \end{cases}$$

$D_i = \{a_i, b_i, c_i\}$ is a balance subset of D.

Considering the set $E = [3(n + \frac{n(n-3)}{2}) + 1, 3n + (\frac{n(n-3)}{2})m^2]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $E_i = \{b^i_j / 1 \leq j \leq m^2 - 3\}$, with

$$b^i_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

E_i is a balance subset of E.

Define a function $h_1 : V(\overline{C_n}[\overline{K_m}]) \to \{A_i, i \in [1, n]\} \subset A$ and label all vertices in every V_i with the elements of A_i. Define a function $h_2 : E(\overline{C_n}[\overline{K_m}]) \to \{A_i, i \in [n+1, (n + \frac{n(n-3)}{2})]\} \bigcup B$ and label all edges in $P_2[\overline{K_m}]_i$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A_{n+i} \bigcup B_i$.

Let $m > 3$ and m be odd. Consider the set $A^* = [1, m(n + \frac{n(n-3)}{2})]$. Divide A^* to be the two

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sets A and E where $\begin{array}{rcl} A & = & [1, 3(n + \frac{n(n-3)}{2})]; \\ E & = & [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})]. \end{array}$

With the same way for $m = 3$, A is $(n + \frac{n(n-3)}{2})$-balance set and for every $i \in [1, (n + \frac{n(n-3)}{2})]$, A_i is a balance subset of A. Consider the set $E = [3(n + \frac{n(n-3)}{2}) + 1, m(n + \frac{n(n-3)}{2})]$. For every $i \in [1, (n + \frac{n(n-3)}{2})]$, $E_i = \{e^i_j / 1 \leq j \leq m - 3\}$, where

$$e^i_j = \begin{cases} 3(n + \frac{n(n-3)}{2}) + (n + \frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ 3(n + \frac{n(n-3)}{2}) + 1 + (n + \frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

$E_i = \{e^i_j / 1 \leq j \leq m - 3\}$ is a balance subset of E. Considering the set $M = [m(n + \frac{n(n-3)}{2}) + 1, m^2(n + \frac{n(n-3)}{2}) + mn]$. For every $i \in [1, \frac{n(n-3)}{2}]$, $M_i = \{m^i_j / 1 \leq j \leq m^2 - m\}$, where
$$m^i_j = \begin{cases} m(n + \frac{n(n-3)}{2}) + (\frac{n(n-3)}{2})(j-1) + i, & \text{if j is odd;} \\ m(n + \frac{n(n-3)}{2}) + 1 + (\frac{n(n-3)}{2})j - i, & \text{if j is even.} \end{cases}$$

is a balance subset of M. Define a function $q_1 : V(\overline{C_n}[\overline{K_m}]) \rightarrow \{A^*_i = A_i \bigcup E_i, i \in [1, n]\} \subset A^*$ and label all vertices in every V_i with the elements of $\{A^*_i, i \in [1, n]\}$.

Define a function $q_2 : E(\overline{C_n}[\overline{K_m}]) \rightarrow \{A^*_{n+i} = A_{n+i} \bigcup E_{n+i}\} \bigcup M$ and label all edges in every $P_2[\overline{K_m}]_i$, $i \in [1, \frac{n(n-3)}{2}]$ with the elements of $A^*_{n+i} \bigcup M_i$.

Since for all $i \in [1, \frac{n(n-3)}{6}]$, $(q_1 + q_2)(P_4[\overline{K_m}]_i) = 7 \sum A^*_i + 3 \sum M_i$ then $\overline{C_n}[\overline{K_m}]$ has $P_4[\overline{K_m}]$-magic decomposition.

Now let $n \equiv 9(mod 12)$. From Lemma 2.2 we have that for $n \equiv 9(mod 12)$, $P_4[\overline{K_m}] | \overline{C_n}[\overline{K_m}]$. Now, let m be even. Do the vertex labeling steps and edge labeling steps such in **case 1** in Theorem 2.1. Because $\forall i \in [1, \frac{n(n-3)}{6}]$, $(f_1 + g)(P_4[\overline{K_{mi}}) = 4 \sum Z_i + 3 \sum X_i$ then $\overline{C_n}[\overline{K_m}]$ have $P_4[\overline{K_m}]$-magic decomposition. Suppose m is odd. Do the vertex labeling steps and edge labeling steps such in **case 2** of Theorem 2.1. Since for all $i \in [1, \frac{n(n-3)}{6}]$, $(f_2 + h)(P_4[\overline{K_m}]_i) = 3 \sum Y_i + 2 \sum P^*_i$ and $(f_3 + h)(P_4[\overline{K_m}]_i) = 3(\sum W_i + \sum X_i) + 2 \sum P^*_i$ then $\overline{C_n}[\overline{K_m}]$ has $P_4[\overline{K_m}]$-magic decomposition.

Now let $n \equiv 0(mod 12)$ and m be even. Clearly from Lemma 2.2 that for $n \equiv 0(mod 12)$, $P_4[\overline{K_m}] | \overline{C_n}[\overline{K_m}]$. Do the vertex labeling steps and edge labeling steps such in **case 1** of Theorem 1. Because $\forall i \in [1, \frac{n(n-3)}{6}]$, $(f_1 + g)(P_4[\overline{K_{mi}}) = 4 \sum Z_i + 3 \sum X_i$ then $\overline{C_n}[\overline{K_m}]$ have $P_4[\overline{K_m}]$-magic decomposition. □

Lemma 2.3. $P_{n-2}[\overline{K_m}] | \overline{C_n}[\overline{K_m}]$ *if and only if* $n \equiv 0(mod 2)$

Proof. $(\Rightarrow)$ Suppose $\overline{C_n}$ where $n \equiv 1(mod 2)$ are P_{n-2}-*decomposable* graphs, then
$$\begin{array}{rcl} \frac{|E(\overline{C_n})|}{3} & = & \frac{(2k+1)(2k-2)/(2)}{2k-2}, s \in Z^+ \\ & = & \frac{2k+1}{2} \\ & = & k + \frac{1}{2} \notin Z^+. \end{array}$$
(contradiction).

$(\Leftarrow)$ Let $V(\overline{C_n}) = \{v_1, ..., v_{2k}\}$, $k \in Z^+$ and $N(v_i)$=$V(\overline{C_n}) \setminus \{v_{i-1}, v_{i+1}\}$. Do the next steps to decompose $\overline{C_n}$. Choose the path $L_1 = v_1 - v_3 - v_n - v_4 - v_{n-1} - ...$ and let v_1 be the center of the rotation. Rotate $L1$ such that v_1 on v_2, v_3 on v_4, v_n on v_1 and etc. Do the next rotation such that v_1 on v_3,...etc, and continue the process until all edge are used up. □

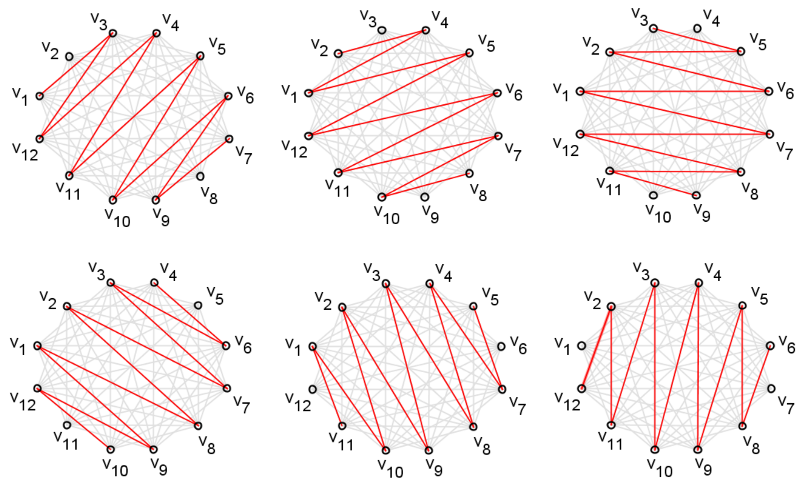

Figure 4. P_9-decomposition of $\overline{C_{12}}$

For example, $\overline{C_{12}}$ in Figure 4 can be decomposed to be 6 P_9-path.

Theorem 2.3. *Let $n > 3$ and $m > 1$. For $n \equiv 2(mod4)$ or ($n \equiv 0(mod4)$ and m is even), $\overline{C_n}[\overline{K_m}]$ have $P_{n-2}[\overline{K_m}]$-magic decomposition.*

Proof. Let $n \equiv 2(mod4)$. From Lemma 2.2 we have that for $n \equiv 2(mod4)$, $P_{n-2}[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$. Now, let m is even. Do the vertex labeling steps and edge labeling steps such in **case 1** of Theorem 2.1. Because of $\forall i \in [1, \frac{n}{2}]$, $(f_1+f_2)(P_{n-2}[\overline{K_{mi}}) = (n-2)m(f_1)+(n-3)m(f_2) = (n-2)(m^2n+m) + (n-3)(\frac{m^2}{2}(2mn+1+\frac{n(n-3)m^2}{2})$. Thus $\overline{C_n}[\overline{K_m}]$ has $P_{n-2}[\overline{K_m}]$-magic decomposition.

Let m be odd. Do the vertex labeling steps and edge labeling steps such in **case 3** of Theorem 2.1. Since for all $i \in [1, \frac{n}{2}]$, $(q_1+q_2)(P_{n-2}[\overline{K_{mi}}) = (2n-5)\sum A_i^* + (n-3)\sum M_i = (2n-5)((2+4n+2n(n-3)+\lceil\frac{2n+n(n-3)}{4}\rceil)+(\frac{m-3}{2})(3(n+\frac{n(n-3)}{2})+1+m(n+\frac{n(n-3)}{2})))+(n-3)(\frac{m^2-m}{2}(m(n+\frac{n(n-3)}{2})+1+m^2(n+\frac{n(n-3)}{2})+mn))$. Thus $\overline{C_n}[\overline{K_m}]$ has $P_4[\overline{K_m}]$-magic decomposition.

Now let $n \equiv 0(mod4)$ and m be even. Clearly from Lemma 2.2 that for $n \equiv 0(mod4)$, $P_{n-2}[\overline{K_m}]|\overline{C_n}[\overline{K_m}]$. Do the vertex labeling steps and edge labeling steps such in **case 1** of Theorem 2.1. Since for all $i \in [1, \frac{n}{2}]$, $(f_1+f_2)(P_{n-2}[\overline{K_{mi}}) = (n-2)m(f_1)+(n-3)m(f_2) = (n-2)(m^2n+m) + (n-3)(\frac{m^2}{2}(2mn+1+\frac{n(n-3)m^2}{2})$. Thus $\overline{C_n}[\overline{K_m}]$ has $P_{n-2}[\overline{K_m}]$-magic decomposition. □

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