$C_4$-decomposition of the tensor product of complete graphs

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Abstract

Let $G$ be a simple and finite graph. A graph is said to be decomposed into subgraphs $H_1$ and $H_2$ which is denoted by $G = H_1 \oplus H_2$, if $G$ is the edge disjoint union of $H_1$ and $H_2$. If $G = H_1 \oplus H_2 \oplus H_3 \oplus \cdots \oplus H_k$, where $H_1, H_2, H_3, \ldots, H_k$ are all isomorphic to $H$, then $G$ is said to be $H$-decomposable. Furthermore, if $H$ is a cycle of length $m$ then we say that $G$ is $C_m$-decomposable and this can be written as $C_m|G$. Where $G \times H$ denotes the tensor product of graphs $G$ and $H$, in this paper, we prove the necessary and sufficient conditions for the existence of $C_4$-decomposition of $K_m \times K_n$. Using these conditions it can be shown that every even regular complete multipartite graph $G$ is $C_4$-decomposable if the number of edges of $G$ is divisible by 4.

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1. Introduction

Let $C_m$, $K_m$ and $K_m - I$ denote cycle of length $m$, complete graph on $m$ vertices and complete graph on $m$ vertices minus a 1-factor respectively. By an $m$-cycle we mean a cycle of length $m$. Let $K_{n,n}$ denote the complete bipartite graph with $n$ vertices in each bipartition set and $K_{n,n} - I$ denote $K_{n,n}$, with a 1-factor removed. All graphs considered in this paper are simple and finite. A graph is said to be decomposed into subgraphs $H_1$ and $H_2$ which is denoted by $G = H_1 \oplus H_2$, if
G is the edge disjoint union of $H_1$ and $H_2$. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $H_1, H_2, \ldots, H_k$ are all isomorphic to $H$, then $G$ is said to be $H$-decomposable. Furthermore, if $H$ is a cycle of length $m$ then we say that $G$ is $C_m$-decomposable and this can be written as $C_m(G)$. A $k$-factor of $G$ is a $k$-regular spanning subgraph. A $k$-factorization of a graph $G$ is a partition of the edge set of $G$ into $k$-factors. A $C_k$-factor of a graph is a 2-factor in which each component is a cycle of length $k$. A resolvable $k$-cycle decomposition (for short $k$-RCD) of $G$ denoted by $C_k||G$, is a 2-factorization of $G$ in which each 2-factor is a $C_k$-factor.

For two graphs $G$ and $H$ their tensor product $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. From this, note that the tensor product of graphs is distributive over edge disjoint union of graphs, that is if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. Now, for $h \in V(H)$, $V(G) \times h = \{(v, h) | v \in V(G)\}$ is called a column of vertices of $G \times H$ corresponding to $h$. Further, for $y \in V(G)$, $y \times V(H) = \{(y, v) | v \in V(H)\}$ is called a layer of vertices of $G \times H$ corresponding to $y$. It is true that $K_m \times K_n$ is isomorphic to the complete bipartite graph $K_{m,n}$ with the edges of a perfect matching removed, i.e. $K_m \times K_n \cong K_{m,n} - I$, where $I$ is a 1-factor of $K_{m,n}$.

The lexicographic product $G \ast H$ of two graphs $G$ and $H$ is the graph having the vertex set $V(G) \times V(H)$, in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if either $g_1, g_2 \in E(G)$; or $g_1 = g_2$ and $h_1, h_2 \in E(H)$.

For very recent works on decomposition of graphs, see [6, 8]. The problem of finding $C_k$-decomposition of $K_{2n+1}$ or $K_{2n} - I$ where $I$ is a 1-factor of $K_{2n}$, is completely settled by Alspach, Gavlas and Šajna in two different papers (see [2, 17]). A generalization to the above complete graph decomposition problem is to find a $C_k$-decomposition of $K_m \ast \overline{K}_n$, which is the complete $m$-partite graph in which each partite set has $n$ vertices. The study of cycle decompositions of $K_m \ast \overline{K}_n$ was initiated by Hoffman et al. [7]. In the case when $p$ is a prime, the necessary and sufficient conditions for the existence of $C_p$-decomposition of $K_m \ast \overline{K}_n$, $p \geq 5$ is obtained by Manikandan and Paulraja in [11, 12, 14]. Billington [3] has studied the decomposition of complete tripartite graphs into cycles of length 3 and 4. Furthermore, Cavenagh and Billington [5] have studied 4-cycle, 6-cycle and 8-cycle decomposition of complete multipartite graphs. Billington et al. [4] have solved the problem of decomposing $(K_m \ast \overline{K}_n)$ into $5$-cycles. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of $C_{2p}$-decomposition of $K_m \ast \overline{K}_n$ is obtained by Smith (see [19]). For a prime $p \geq 3$, it was proved in [20] that $C_{3p}$-decomposition of $K_m \ast \overline{K}_n$ exists if the obvious necessary conditions are satisfied. As the graph $K_m \ast K_n \cong K_m \ast \overline{K}_n - E(nK_m)$ is a proper regular spanning subgraph of $K_m \ast \overline{K}_n$. It is therefore natural to think about the cycle decomposition of $K_m \ast K_n$.

The results in [11, 12, 14] also gives the necessary and sufficient conditions for the existence of a $p$-cycle decomposition, (where $p \geq 5$ is a prime number) of the graph $K_m \ast \overline{K}_n$. In [13] it was shown that the tensor product of two regular complete multipartite graph is Hamilton cycle decomposable. Muthusamy and Paulraja in [15] proved the existence of $C_{kn}$-factorization of the graph $C_k \times K_{m,n}$, where $mn \neq 2(\text{mod } 4)$ and $k$ is odd. Paulraja and Kumar [16] showed that the necessary conditions for the existence of a resolvable $k$-cycle decomposition of tensor product of complete graphs are sufficient when $k$ is even. In a recent work by the present authors, it was proven that the necessary and sufficient conditions for the decomposition of the graph $K_m \times K_n$.
into cycles of length six is that \( m \) or \( n \equiv 1 \) or \( 3 \pmod{6} \) (see [1]).
In this paper, we prove the necessary and sufficient conditions for \( K_m \times K_n, \) where \( m, n \geq 2, \) to have a \( C_4 \)-decomposition. Among other results, here we prove the following main result.

**Theorem 1.1.** For \( m, n \geq 2, \) \( C_4 \mid K_m \times K_n \) if and only if either
1. \( n \equiv 0 \pmod{4} \) and \( m \) is odd,
2. \( m \equiv 0 \pmod{4} \) and \( n \) is odd or
3. \( m \) or \( n \equiv 1 \pmod{4} \)

Let \( \rho \) be a permutation of the vertex set \( V \) of a graph \( G. \) For any subset \( U \) of \( V, \) \( \rho \) acts as a function from \( U \) to \( V \) by considering the restriction of \( \rho \) to \( U. \) If \( H \) is a subgraph of \( G \) with vertex set \( U, \) then \( \rho(H) \) is a subgraph of \( G \) provided that for each edge \( xy \in E(H), \rho(x)\rho(y) \in E(G). \)

Next, we give some existing results on cycle decomposition of complete graphs.

**Theorem 1.2.** [9] Let \( m \) be an odd integer and \( m \geq 3. \) If \( m \equiv 1 \) or \( 3 \pmod{6} \) then \( C_3 \mid K_m. \)

**Theorem 1.3.** [17] Let \( n \) be an odd integer and \( m \) be an even integer with \( 3 \leq m \leq n. \) The graph \( K_n \) can be decomposed into cycles of length \( m \) whenever \( m \) divides the number of edges in \( K_n. \)

Now we have the following lemma, this lemma gives the cycle decomposition of the complete graph \( K_m \) into cycles of length 3 and 4.

**Lemma 1.1.** For \( m \equiv 5 \pmod{6}, \) there exist positive integers \( p \) and \( q \) such that \( K_m \) is decomposable into \( p \) 3-cycles and \( q \) 4-cycles.

**Proof.** Let the vertices of \( K_m \) be \( 0, 1, \ldots, m - 1. \) The 4-cycles are \((i, i+1+2s, i-1, i+2+2s), s = 0, 1, \ldots, (m-i)/2 - 2, i = 1, 3, \ldots, m - 4. \) The 3-cycles are \((m-1, i-1, i), i = 1, 3, \ldots, m - 2. \) Hence the proof.

The following theorem is on the complete bipartite graph minus a 1-factor, it was obtained by Ma et. al [10].

**Theorem 1.4.** [10] Let \( m \) and \( n \) be positive integers. Then there exist an \( m \) cycle system of \( K_{n,n} - I \) if and only if \( n \equiv 1 \pmod{2}, m \equiv 0 \pmod{2}, 4 \leq m \leq 2n \) and \( n(n-1) \equiv 0 \pmod{m} \).

From the theorem above we have the following corollary.

**Corollary 1.1.** The graph \( K_{n,n} - I, \) where \( I \) is a 1-factor of \( K_{n,n} - I \) admits a \( C_4 \) decomposition if and only if \( n \equiv 1 \pmod{4}. \)

The following result is on the complete bipartite graphs.

**Theorem 1.5.** [18] The complete bipartite graph \( K_{a,b} \) can be decomposed into cycles of length \( 2k \) if and only if \( a \) and \( b \) are even, \( a \geq k, b \geq k \) and \( 2k \) divides \( ab. \)
2. $C_4$ Decomposition of $C_m \times K_n$

We begin this section with the following lemma.

**Lemma 2.1.** $C_4|C_3 \times K_4$.

*Proof.* Following from the definition of the tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}, U^2 = \{u_2, v_2, w_2\}, ..., U^4 = \{u_4, v_4, w_4\}$ form the partite sets of vertices in the product $C_3 \times K_4$. For $1 \leq i, j \leq 4$, surely $U^i \cup U^j$, $i \neq j$ induces a $K_{3,3} - I$, where $I$ is a 1-factor of $K_{3,3}$.

A $C_4$ decomposition of $C_3 \times K_4$ is given below:

\[
\{u_1, v_4, u_2, w_3\}, \{u_1, v_3, u_4, w_2\}, \{u_1, v_2, u_3, w_4\}, \{u_2, v_3, w_2, v_1\}, \{u_3, v_1, w_3, v_4\}, \{u_2, w_1, v_3, w_4\}, \{u_3, w_2, v_4, w_1\}, \{u_4, v_1, w_4, v_2\} \text{ and } \{u_4, w_1, v_2, w_3\}
\]

Next, we have the following lemma which follows from Lemma 2.1.

**Lemma 2.2.** $C_4|C_3 \times K_5$.

*Proof.* Suppose we fix the 4-cycles already given in Lemma 2.1, clearly the graph which remains after removing the edges of $C_3 \times K_4$ from $C_4 \times K_5$ can be decomposed into 3 copies of $K_{2,4}$. Now, by Theorem 1.5 the graph $K_{2,4}$ can be decomposed into cycles of length 4. Hence $C_4|C_3 \times K_5$.

The following theorem is an extension of Lemma 2.1 and Lemma 2.2.

**Theorem 2.1.** $C_4|C_3 \times K_n$ if and only if $n \equiv 0 \text{ or } 1 \pmod{4}$.

*Proof.* Suppose that $C_4|C_3 \times K_n$. The graph $C_3 \times K_n$ has $3n(n - 1)$ edges. For $C_4|C_3 \times K_n$ it implies that $n(n - 1) \equiv 0 \pmod{4}$. Hence $n \equiv 0 \text{ or } 1 \pmod{4}$.

Following the definition of tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}, U^2 = \{u_2, v_2, w_2\}, ..., U^n = \{u_n, v_n, w_n\}$ form the partite sets of vertices in the product $C_3 \times K_n$. For $1 \leq i, j \leq n$, surely $U^i \cup U^j$, $i \neq j$ induces a $K_{3,3} - I$, where $I$ is a 1-factor of $K_{3,3}$.

Next, we prove the sufficiency in two cases.

**Case 1.** Whenever $n \equiv 0 \pmod{4}$. Let $n = 4t$ where $t \geq 1$.

Next we note that $C_3 \times K_n \cong (C_3 \times K_4) + (C_3 \times K_4) + ... + (C_3 \times K_4) + H^*$, $H^*$ is the graph containing the edges of $C_3 \times K_n$ which are not covered by these $t$ copies of $C_3 \times K_4$.

By Lemma 2.1 the product $C_3 \times K_4$ admits a $C_4$-decomposition. Furthermore, we define the set $U = \{u^1, u^2, ..., u^p\}, V = \{v^1, v^2, ..., v^p\}$ and $W = \{w^1, w^2, ..., w^p\}$ where $p = n/4$ and for $j = 1, 2, ..., p$, $u^i = \{u_1|i4j - 3 \leq i \leq 4j\}$, $v^i = \{v_1|i4j - 3 \leq i \leq 4j\}$ and $w^i = \{w_1|i4j - 3 \leq i \leq 4j\}$.

Now, $H^*$ is decomposable into graphs isomorphic to $K_{4,4n-4}$. Indeed, the $K_{4,4n-4}$ graphs in the decomposition of $H^*$ are induced by $(u^i \cup v^i \cup w^i \cup ... \cup u^p) \setminus v^i$, $(u^i \cup w^i \cup w^2 \cup ... \cup w^p) \setminus w^i$ and $(v^i \cup u^i \cup w^2 \cup ... \cup u^p) \setminus w^i, i = 1, 2, ..., p$. By Theorem 1.5 $C_4|K_{4,4n-4}$. Therefore we have decomposed $C_3 \times K_n$ into 4-cycles when $n \equiv 0 \pmod{4}$.

**Case 2.** Whenever $n \equiv 1 \pmod{4}$. Let $n = 4t + 1$ where $t \geq 1$.

By removing $U^1$, we obtain a copy of $C_3 \times K_{n-1}$, so we may apply Case 1. The remaining structure can be decomposed into $3K_{2,4t}$ and by Theorem 1.5 $C_4|K_{2,4t}$. Therefore $C_4|C_3 \times K_n$ when $n \equiv 1 \pmod{4}$.
Next, we establish the following lemma.

**Lemma 2.3.** For all \( n \geq 3 \), \( C_4|C_4 \times K_n \).

**Proof.** From the definition of tensor product of graphs, let \( U^1 = \{ u_1, v_1, w_1, x_1 \} \), \( U^2 = \{ u_2, v_2, w_2, x_2 \} \), \ldots, \( U^n = \{ u_n, v_n, w_n, x_n \} \) form the partite sets of vertices in the product \( C_4 \times K_n \). Also, for \( 1 \leq i, j \leq n \) and \( i \neq j \), \( U^i \cup U^j \) induces \( K_{4,4} - 2I \), where \( I \) is a 1-factor of \( K_{4,4} \). Now, each set \( U^i \cup U^j \) is isomorphic to \( K_{4,4} - 2I \). But \( K_{4,4} - 2I \) admits a 4-cycle decomposition. Hence the proof.

Furthermore, we quote the following result on decomposition of the tensor product of graphs into cycles of odd length.

**Lemma 2.4.** [12] For \( k \geq 1 \) and \( m \geq 3 \), \( C_{2k+1}|C_{2k+1} \times K_m \)

The next lemma is an extension of Lemma 2.3 and Lemma 2.4.

**Lemma 2.5.** For \( m \geq 3 \) and \( n \geq 2 \), \( C_m|C_m \times K_n \)

**Proof.** We shall split the proof of this lemma into two cases.

**Case 1.** When \( m = 2k + 1 \), \( k \geq 1 \)

The proof of this case is immediate from Lemma 2.4.

**Case 2.** When \( m = 2k \), \( k \geq 2 \)

Following from the definition of tensor product of graphs, let \( U_1 = \{ u^1_1, u^2_1, u^3_1, \ldots, u^m_1 \} \), \( U_2 = \{ u^1_2, u^2_2, u^3_2, \ldots, u^m_2 \} \), \ldots, \( U_n = \{ u^1_n, u^2_n, u^3_n, \ldots, u^m_n \} \) form the partite sets of vertices in the product \( C_m \times K_n \). Now, for \( 1 \leq i, j \leq n \) and \( i \neq j \), the subgraph induced by \( U_i \cup U_j \) is isomorphic to \( K_{m,m} - (m - 2)I \), where \( I \) is a 1-factor of \( K_{m,m} \). But \( K_{m,m} - (m - 2)I \) admits an \( m \)-cycle decomposition. Hence the proof.

3. **Proof of the Main Theorem**

**Proof of Theorem 1.1.** Assume that \( C_4|K_m \times K_n \), for some \( m \) and \( n \) with \( 2 \leq m, n \). Then every vertex of \( K_m \times K_n \) has even degree and 4 divides the number of edges of \( K_m \times K_n \). These two conditions translates to \( (m - 1)(n - 1) \) being even and \( 8|mn(m - 1)(n - 1) \) respectively. Hence by the first fact, \( m \) or \( n \) has to be odd, i.e. has to be congruent to 1 or 3 or 5 \((\text{mod } 6)\). The second condition is satisfied precisely when one of the following holds.

1. \( n \equiv 0 \pmod{4} \) and \( m \) is odd,
2. \( m \equiv 0 \pmod{4} \) and \( n \) is odd, or
3. \( m \) or \( n \equiv 1 \pmod{4} \).
Next we proceed to prove the sufficiency in two cases.

**Case 1.** Since the tensor product is commutative, we may assume that \( m \) is odd and so \( m \equiv 1 \text{ or } 3 \text{ or } 5 \pmod{6} \). Suppose that \( n \equiv 0 \pmod{4} \).

**Subcase 1.** Let \( m \equiv 1 \text{ or } 3 \pmod{6} \)

Now since \( m \equiv 1 \text{ or } 3 \pmod{6} \) it implies that by Theorem 1.2 \( C_3|K_m \). Therefore, the graph \( K_m \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)) \). But \( n \equiv 0 \pmod{4} \) therefore by Theorem 2.1 we have that \( C_4|C_3 \times K_n \). Hence \( C_4|K_m \times K_n \).

**Subcase 2.** Let \( m \equiv 5 \pmod{6} \)

By Lemma 1.1, there exist positive integers \( p \) and \( q \) such that \( K_m \) is decomposable into \( p \) 3-cycles and \( q \) 4-cycles. Hence \( K_m \times K_n \) has a decomposition into \( p \) copies of \( C_3 \times K_n \) and \( q \) copies of \( C_4 \times K_n \). By Theorem 2.1 \( C_4|C_3 \times K_n \) and also Lemma 2.3 shows that \( C_4|C_4 \times K_n \). Hence \( C_4|K_m \times K_n \). This completes the proof. \( \square \)

Lastly, we draw our conclusion by the following remark.

**Remark 3.1.** The product \( K_m \times K_n \) can also be viewed as an even regular complete multipartite graph. So by the conditions given in Theorem 1.1 we have that every even regular complete multipartite graph \( G \) is \( C_4 \)-decomposable if the number of edges of \( G \) is divisible by 4.

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