Multidesigns for the graph pair formed by the 6-cycle and 3-prism

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Abstract
Given two graphs $G$ and $H$, a $(G, H)$-multidecomposition of $K_n$ is a partition of the edges of $K_n$ into copies of $G$ and $H$ such that at least one copy of each is used. We give necessary and sufficient conditions for the existence of $(C_6, \overline{C}_6)$-multidecomposition of $K_n$ where $C_6$ denotes a cycle of length 6 and $\overline{C}_6$ denotes the complement of $C_6$. We also characterize the cardinalities of leaves and paddings of maximum $(C_6, \overline{C}_6)$-multipackings and minimum $(C_6, \overline{C}_6)$-multicoverings, respectively.

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1. Introduction

Let $G$ and $H$ be graphs. Denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. A $G$-decomposition of $H$ is a partition of $E(H)$ into a set of edge-disjoint subgraphs of $H$ each of which is isomorphic to $G$. Graph decompositions have been extensively studied. This is particularly true for the case where $H \cong K_n$, see [2] for a recent survey. A $G$-decomposition of $K_n$ is sometimes referred to as a $G$-design of order $n$. As an extension of a graph decomposition we can permit more than one graph, up to isomorphism, to appear in the partition. A $(G, H)$-multidecomposition of $K_n$ is a partition of $E(K_n)$ into a set of edge-disjoint subgraphs each of
which is isomorphic to either $G$ or $H$, and at least one copy of $G$ and one copy of $H$ are elements of the partition. When a $(G, H)$-multidecomposition of $K_n$ does not exist, we would like to know how “close” we can get. More specifically, define a $(G, H)$-multipacking of $K_n$ to be a collection of edge-disjoint subgraphs of $K_n$ each of which is isomorphic to either $G$ or $H$ such that at least one copy of each is present. The set of edges in $K_n$ that are not used as copies of either $G$ or $H$ in the $(G, H)$-multipacking is called the leave of the $(G, H)$-multipacking. Similarly, define a $(G, H)$-multicovering of $K_n$ to be a partition of the multiset of edges formed by $E(K_n)$ where some edges may be repeated into edge-disjoint copies of $G$ and $H$ such that at least one copy of each is present. The multiset of repeated edges is called the padding of the partition. When a $(G, H)$-multipacking or $(G, H)$-multicovering does not exist, we would like to know how “close” we can get. More specifically, define a $(G, H)$-multipacking of $K_n$ to be a collection of edge-disjoint subgraphs of $K_n$ each of which is isomorphic to either $G$ or $H$ such that at least one copy of each is present. The set of edges in $K_n$ that are not used as copies of either $G$ or $H$ in the $(G, H)$-multipacking is called the leave of the $(G, H)$-multipacking.

Theorem 1.1. The complete graph $K_n$ admits a $(C_6, \overline{C}_6)$-multidecomposition of $K_n$ if and only if $n \equiv 0, 1 \pmod{3}$ with $n \geq 6$, except $n \in \{7, 9, 10\}$.

Theorem 1.2. For each $n \equiv 2 \pmod{3}$ with $n \geq 8$, a maximum $(C_6, \overline{C}_6)$-multipacking of $K_n$ has a leave of cardinality 1. Furthermore, a maximum $(C_6, \overline{C}_6)$-multipacking of $K_7$ has a leave of cardinality 6, and a maximum $(C_6, \overline{C}_6)$-multipacking of either $K_9$ or $K_{10}$ has a leave of cardinality 3.

Theorem 1.3. For each $n \equiv 2 \pmod{3}$ with $n \geq 8$, a minimum $(C_6, \overline{C}_6)$-multicovering of $K_n$ has a padding of cardinality 2. Furthermore, a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ has a padding of cardinality 6, and a minimum $(C_6, \overline{C}_6)$-multicovering of either $K_9$ or $K_{10}$ has a padding of cardinality 2.

Let $G$ and $H$ be vertex-disjoint graphs. The join of $G$ and $H$, denoted $G \vee H$, is defined to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{u, v\} : u \in V(G), v \in V(H)$. We use the shorthand notation $\bigvee_{i=1}^{t} G_i$ to denote $G_1 \vee G_2 \vee \cdots \vee G_t$, and when $G_i \cong G$ for all $1 \leq i \leq t$ we write $\bigvee_{i=1}^{t} G$. For example, $K_{12} \cong \bigvee_{i=1}^{4} K_3$. 

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For notational convenience, let \((a, b, c, d, e, f)\) denote the copy of \(C_6\) with vertex set \(\{a, b, c, d, e, f\}\) and edge set \(\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{a, f\}\}\), as seen in Figure 1. Let \([a, c; d, e, f]\) denote the copy of \(\overline{C_6}\) with vertex set \(\{a, b, c, d, e, f\}\) and edge set

\[
\{\{a, b\}, \{b, c\}, \{a, c\}, \{d, e\}, \{e, f\}, \{d, f\}, \{a, d\}, \{b, e\}, \{c, f\}\}.
\]

Figure 1. Labeled copies of \(C_6\) and \(\overline{C_6}\), denoted by \((a, b, c, d, e, f)\) and \([a, c; d, e, f]\), respectively.

Next, we state some known results on graph decompositions that will help us prove our main result. Sotteau’s theorem gives necessary and sufficient conditions for complete bipartite graphs (denoted by \(K_{m,n}\) when the partite sets have cardinalities \(m\) and \(n\)) to decompose into even cycles of fixed length. Here we state the result only for cycle length 6.

**Theorem 1.4** (Sotteau [5]). A \(C_6\)-decomposition of \(K_{m,n}\) exists if and only if \(m \geq 4, n \geq 4, m\) and \(n\) are both even, and 6 divides \(mn\).

Another celebrated result in the field of graph decompositions is that the necessary conditions for a \(C_k\)-decomposition of \(K_n\) are also sufficient. Here we state the result only for \(k = 6\).

**Theorem 1.5** (Šajna [4]). Let \(n\) be a positive integer. A \(C_6\)-decomposition of \(K_n\) exists if and only if \(n \equiv 1, 9 \pmod{12}\).

The necessary and sufficient conditions for a \(\overline{C_6}\)-decomposition of \(K_n\) are also known, and stated in the following theorem.

**Theorem 1.6** (Kang et al. [3]). Let \(n\) be a positive integer. A \(\overline{C_6}\)-decomposition of \(K_n\) exists if and only if \(n \equiv 1 \pmod{9}\).

2. Multidecompositions

We first establish the necessary conditions for a \((C_6, \overline{C_6})\)-multidecomposition of \(K_n\).

**Lemma 2.1.** If a \((C_6, \overline{C_6})\)-multidecomposition of \(K_n\) exists, then

1. \(n \geq 6\), and
2. \(n \equiv 0, 1 \pmod{3}\).
Proof. Assume that a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) exists. It is clear that condition (1) holds. Considering that the edges of \(K_n\) are partitioned into subgraphs isomorphic to \(C_6\) and \(\overline{C}_6\), we have that there exist positive integers \(x\) and \(y\) such that \(\binom{n}{2} = 6x + 9y\). Hence, 3 divides \(\binom{n}{2}\), which implies \(n \equiv 0, 1 \pmod{3}\), and condition (2) follows.

2.1. Small examples of multidecompositions

In this section we present various non-existence and existence results for \((C_6, \overline{C}_6)\)-multidecompositions of small orders. The existence results will help with our general constructions.

2.1.1. Non-existence results

The necessary conditions for the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_n\) fail to be sufficient in exactly three cases, namely \(n = 7, 9, 10\). We will now establish the non-existence of \((C_6, \overline{C}_6)\)-multidecompositions of \(K_n\) for these cases.

Lemma 2.2. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\), call it \(G\). There must exist positive integers \(x\) and \(y\) such that \(\binom{7}{2} = 21 = 6x + 9y\). The only solution to this equation is \((x, y) = (2, 1)\); therefore, \(G\) must contain exactly one copy of \(\overline{C}_6\). However, upon examining the degree of each vertex contained in the single copy of \(\overline{C}_6\) we see that there must exist a non-negative integer \(p\) such that \(6 = 2p + 3\). This is a contradiction. Thus, a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) cannot exist.

Lemma 2.3. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\), call it \(G\). There must exist positive integers \(x\) and \(y\) such that \(\binom{9}{2} = 36 = 6x + 9y\). The only solution to this equation is \((x, y) = (3, 2)\); therefore, \(G\) must contain exactly two copies of \(\overline{C}_6\).

Turning to the degrees of the vertices in \(K_9\), we have that there must exist positive integers \(p\) and \(q\) such that \(8 = 2p + 3q\). The only possibilities are \((p, q) \in \{(4, 0), (1, 2)\}\). Note that \(K_6\) does not contain two edge-disjoint copies of \(\overline{C}_6\). Since \(G\) contains exactly two copies of \(\overline{C}_6\), there must exist at least one vertex \(a \in V(K_9)\) that is contained in exactly one copy of \(\overline{C}_6\). However, this contradicts the fact that vertex \(a\) must be contained in either 0 or 2 copies of \(\overline{C}_6\). Thus, a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_9\) cannot exist.

Lemma 2.4. A \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{10}\) does not exist.

Proof. Assume the existence of a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{10}\), call it \(G\). There must exist positive integers \(x\) and \(y\) such that \(\binom{10}{2} = 45 = 6x + 9y\). Thus, \((x, y) \in \{(6, 1), (3, 3)\}\); therefore, \(G\) must contain at least one copy of \(\overline{C}_6\). However, if \(G\) consists of exactly one copy of \(\overline{C}_6\), then the vertices of \(K_{10}\) which are not included in this copy would have odd degrees remaining after the removal of the copy of \(\overline{C}_6\). Thus, the case where \((x, y) = (6, 1)\) is impossible.

Upon examining the degree of each vertex in \(K_{10}\), we see that there must exist positive integers \(p\) and \(q\) such that \(9 = 2p + 3q\). The only solutions to this equation are \((p, q) \in \{(3, 1), (0, 3)\}\). From the above argument, we know that \(G\) contains exactly 3 copies of \(\overline{C}_6\), say \(A, B,\) and \(C\). Let
Example 1. \(K_{13}\) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Let \(V(K_{13}) = \{1, 2, \ldots, 13\}\). The following is a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{13}\).

\[
\begin{align*}
\{ & [1, 2, 3; 7, 9, 8], [1, 4, 5; 9, 12, 10], [3, 4, 6; 7, 11, 10], [2, 5, 6; 8, 12, 11] \\
\cup & \{(13, 1, 6, 8, 5, 11), (13, 2, 4, 7, 6, 12), (13, 3, 5, 9, 4, 10), (13, 7, 12, 3, 9, 6),
\quad (13, 8, 10, 2, 7, 5), (13, 9, 11, 1, 8, 4), (1, 10, 3, 11, 2, 12)\}\end{align*}
\]

Example 2. \(K_{15}\) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Let \(V(K_{15}) = \{1, 2, \ldots, 15\}\). The following is a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{15}\).

\[
\begin{align*}
\{ & [1, 5, 10; 6, 8, 12], [4, 8, 13; 9, 11, 15], [7, 11, 1; 12, 14, 3], [10, 14, 4; 15, 2, 6],
\quad [13, 2, 7; 3, 5, 9] \\
\cup & \{(1, 12, 11, 13, 5, 15), (4, 15, 14, 1, 8, 3), (7, 3, 2, 4, 11, 6), (10, 6, 5, 7, 14, 9),
\quad (13, 9, 8, 10, 2, 12), (1, 2, 11, 3, 6, 13), (4, 5, 14, 6, 9, 1), (7, 8, 2, 9, 12, 4),
\quad (10, 11, 5, 12, 15, 7), (13, 14, 8, 15, 3, 10)\}\end{align*}
\]

Example 3. \(K_{19}\) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Let \(V(K_{19}) = \{1, 2, \ldots, 19\}\). The following is a \((C_6, \overline{C}_6)\)-multidecomposition of \(K_{19}\).

\[
\begin{align*}
\{ & [2, 11, 14; 17, 4, 18], [3, 12, 15; 18, 5, 19], [4, 13, 16; 19, 6, 11], [5, 14, 17; 11, 7, 12],
\quad [6, 15, 18; 12, 8, 13], [7, 16, 19; 13, 9, 14], [8, 17, 11; 14, 10, 15], [9, 18, 12; 15, 2, 16],
\quad [10, 19, 13; 16, 3, 17] \\
\cup & \{(2, 12, 14, 3, 11, 1), (3, 13, 15, 4, 12, 1), (4, 14, 16, 5, 13, 1), (5, 15, 17, 6, 14, 1),
\quad (6, 16, 18, 7, 15, 1), (7, 17, 19, 8, 16, 1), (8, 18, 11, 9, 17, 1), (9, 19, 12, 10, 18, 1),
\quad (10, 11, 13, 2, 19, 1), (2, 3, 10, 4, 9, 5), (2, 6, 8, 7, 3, 4), (2, 7, 4, 5, 3, 8),
\quad (2, 10, 8, 4, 6, 9), (3, 6, 10, 5, 7, 9), (5, 6, 7, 10, 9, 8)\}\end{align*}
\]
2.2. General constructions for multidecompositions

Lemma 2.5. If \( n \equiv 0 \pmod{6} \) with \( n \geq 6 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x \) for some integer \( x \geq 1 \). Note that \( K_{6x} \cong \bigvee_{i=1}^{x} K_6 \). On each copy of \( K_6 \) place a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_6 \). The remaining edges form edge-disjoint copies of \( K_{6,6} \), which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Lemma 2.6. If \( n \equiv 1 \pmod{6} \) with \( n \geq 13 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 1 \) for some integer \( x \geq 2 \). The proof breaks into two cases.

Case 1: \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+1} \cong K_1 \lor (\bigvee_{i=1}^{k} K_{12}) \). Each of the \( k \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admits a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of \( K_{12,12} \), which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Case 2: \( x = 2k + 1 \) for some integer \( k \geq 2 \). Notice that \( K_{12k+7} \cong K_1 \lor K_6 \lor (\bigvee_{i=1}^{k} K_{12}) \). The single copy of \( K_{19} \) formed by \( K_1 \lor K_6 \lor K_{12} \) admits a \((C_6, \overline{C}_6)\)-multidecomposition by Example 3. The remaining \( k - 1 \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) each admit a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge-disjoint copies of either \( K_{6,12} \) or \( K_{12,12} \). Both of these graphs admit \( C_6 \)-decompositions by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multidecomposition of \( K_n \).

Lemma 2.7. If \( n \equiv 3 \pmod{6} \) with \( n \geq 15 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 3 \) for some integer \( x \geq 2 \). The proof breaks into two cases.

Case 1: \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+3} \cong K_1 \lor K_{14} \lor (\bigvee_{i=1}^{k-1} K_{12}) \). The remainder of the proof is similar to the proof of Case 1 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_{12,12} \), and \( K_{12,14} \), as well as \((C_6, \overline{C}_6)\)-multidecompositions of \( K_{13} \) and \( K_{15} \).

Case 2: \( x = 2k + 1 \) for some integer \( k \geq 1 \). Notice that \( K_{12k+9} \cong K_1 \lor K_8 \lor (\bigvee_{i=1}^{k} K_{12}) \). The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_9 \) (which exists by Theorem 1.5), \( K_{8,12} \), and \( K_{12,12} \), as well as a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_{13} \).

Lemma 2.8. If \( n \equiv 4 \pmod{6} \) with \( n \geq 16 \), then \( K_n \) admits a \((C_6, \overline{C}_6)\)-multidecomposition.

Proof. Let \( n = 6x + 4 \) where \( x \geq 2 \) is an integer. Note that \( K_{6x+4} \cong K_{10} \lor (\bigvee_{i=1}^{x-1} K_6) \). The remainder of the proof is similar to the proof of Case 2 of Lemma 2.6 where the ingredients required are \( C_6 \)-decompositions of \( K_{6,6} \) and \( K_{6,10} \), a \( \overline{C}_6 \)-decomposition of \( K_{10} \) (which exists by Theorem 1.6), as well as a \((C_6, \overline{C}_6)\)-multidecomposition of \( K_6 \).

Combining Lemmas 2.5, 2.6, 2.7, and 2.8, we have proven Theorem 1.1.
3. Maximum Multipackings

Now we turn our attention to \((C_6, \overline{C}_6)\)-multipackings in the cases where \((C_6, \overline{C}_6)\)-multidecompositions do not exist.

3.1. Small examples of maximum multipackings

Example 4. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_7\) has a leave of cardinality 6.

Note that the number of edges used in a \((C_6, \overline{C}_6)\)-multipacking of any graph must be a multiple of 3, since \(\gcd(6, 9) = 3\). Since no \((C_6, \overline{C}_6)\)-multidecomposition of \(K_7\) exists the next possibility is a leave of cardinality 3. However, the equation \(18 = 6x + 9y\) has no positive integer solutions. Thus, the minimum possible cardinality of a leave is 6. Let \(V(K_7) = \{1, ..., 7\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_7\), with leave \(\{1, 7\}, \{2, 7\}, \{3, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\}\).

\[
\{(1, 3, 5; 4, 6, 2), (1, 2, 3, 4, 5, 6)\}
\]

Example 5. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_8\) has a leave of cardinality 1.

Let \(V(K_8) = \{1, ..., 8\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_8\), with leave \(\{3, 6\}\).

\[
\{(2, 5, 7; 4, 1, 8), (1, 2, 3, 4, 5, 6), (1, 3, 5, 8, 6, 7), (3, 8, 2, 6, 4, 7)\}
\]

Example 6. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_9\) has a leave of cardinality 3.

Let \(V(K_9) = \{1, ..., 9\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_9\), with leave \(\{2, 4\}, \{2, 9\}, \{4, 9\}\).

\[
\{(1, 2, 3; 6, 5, 4), [1, 4, 7; 9, 8, 3], [2, 6, 8; 7, 9, 5], (1, 5, 3, 6, 7, 8)\}
\]

Example 7. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) has a leave of cardinality 3.

A \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) with a leave of cardinality 3 can be obtained by starting with a \(\overline{C}_6\)-decomposition of \(K_{10}\). Then remove three vertex-disjoint edges from one copy of \(\overline{C}_6\), forming a \(C_6\). This gives us the desired \((C_6, \overline{C}_6)\)-multipacking of \(K_{10}\) where the three removed edges form the leave.

Example 8. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{11}\) has a leave of cardinality 1.

Let \(V(K_{11}) = \{1, ..., 11\}\). The following is a \((C_6, \overline{C}_6)\)-multipacking of \(K_{11}\), with leave \(\{1, 2\}\).

\[
\{(1, 7, 10; 9, 6, 3), [1, 5, 6; 4, 10, 2], [2, 5, 7; 11, 8, 4], [1, 3, 11; 8, 2, 9]\}
\cup\{(3, 4, 9, 10, 6, 8), (4, 5, 9, 7, 11, 6), (3, 5, 11, 10, 8, 7)\}
\]

Example 9. A maximum \((C_6, \overline{C}_6)\)-multipacking of \(K_{17}\) has a leave of cardinality 1.
Let $V(K_{17}) = \{1, ..., 17\}$. The following is a $(C_6, \overline{C_6})$-multipacking of $K_{17}$, with leave $\{1, 10\}$:

$$\{ [2, 3, 5; 7, 8, 1], [3, 6, 4; 9, 8, 10], [2, 4, 9; 6, 5, 7] \} \cup \{ (2, 12, 5, 10, 11, 14), (2, 10, 17, 4, 13, 11), (4, 7, 13, 14, 5, 15), (4, 11, 15, 8, 16, 12), (1, 15, 14, 16, 5, 17), (3, 12, 11, 17, 6, 15), (1, 2, 16, 7, 14, 4), (2, 13, 5, 8, 14, 17), (7, 15, 10, 13, 9, 17), (1, 13, 6, 9, 11, 16), (1, 9, 12, 7, 3, 11), (3, 10, 12, 8, 4, 16), (3, 13, 16, 6, 12, 14), (2, 8, 13, 17, 12, 15), (6, 10, 16, 15, 9, 14), (5, 9, 16, 17, 8, 11), (1, 3, 17, 15, 13, 12), (1, 6, 11, 7, 10, 14) \}$$

### 3.2. General Constructions of maximum multipackings

**Lemma 3.1.** If $n \equiv 2 \pmod{6}$ with $n \geq 14$, then $K_n$ admits a $(C_6, \overline{C_6})$-multipacking with leave cardinality 1.

**Proof.** Let $n = 6x + 2$ for some integer $x \geq 2$. Notice that $K_{6x + 2} \cong K_2 \lor (\bigvee_{i=1}^{x} K_6)$. Let $\{u, v\} = V(K_2)$. Each of the $x$ copies of $K_8$ formed by $K_2 \lor K_6$ admit a $(C_6, \overline{C_6})$-multipacking with leave cardinality 1 by Example 5. Note that we can always choose the leave edge to be $\{u, v\}$ in each of these multipackings. The remaining edges form edge disjoint copies of $K_{6,6}$, each of which admits a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C_6})$-multipacking of $K_n$. 

**Lemma 3.2.** If $n \equiv 5 \pmod{6}$ with $n \geq 11$, then $K_n$ admits a $(C_6, \overline{C_6})$-multipacking with leave cardinality 1.

**Proof.** Let $n = 6x + 5$ for some integer $x \geq 1$.

**Case 1:** $x = 2k$ for some integer $k \geq 1$. Notice that $K_{12k+5} \cong K_1 \lor K_{16} \lor (\bigvee_{i=1}^{k-1} K_{12})$. Each of the $k - 1$ copies of $K_{13}$ formed by $K_1 \lor K_{12}$ admit a $(C_6, \overline{C_6})$-multidecomposition by Example 1. The copy of $K_{17}$ formed by $K_1 \lor K_{16}$ admits a $(C_6, \overline{C_6})$-multipacking with leave of cardinality 1 by Example 9. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{12,16}$, each of which admits a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C_6})$-multipacking of $K_n$.

**Case 2:** $x = 2k + 1$ for some integer $k \geq 1$. Notice that $K_{12k+11} \cong K_1 \lor K_{10} \lor (\bigvee_{i=1}^{k} K_{12})$. On each of the $k$ copies of $K_{13}$ formed by $K_1 \lor K_{12}$ admit a $(C_6, \overline{C_6})$-multidecomposition by Example 1. The copy of $K_{11}$ formed by $K_1 \lor K_{10}$ admits a $(C_6, \overline{C_6})$-multipacking with leave of cardinality 1 by Example 8. The remaining edges form edge disjoint copies of $K_{12,12}$ or $K_{10,12}$, each of which admits a $C_6$-decomposition by Theorem 1.4. Thus, we obtain the desired $(C_6, \overline{C_6})$-multipacking of $K_n$. 

Combining Lemmas 3.1 and 3.2 along with Examples 4, 6, and 7 we have proven Theorem 1.2.

### 4. Minimum Multicoverings

Now we turn our attention to minimum $(C_6, \overline{C_6})$-multicoverings in the cases where $(C_6, \overline{C_6})$-multidecompositions do not exist.
4.1. Small examples of minimum multicoVERings

Example 10. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ has a padding of cardinality 6.

We first rule out the possibility of a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$ with a padding of cardinality 3. The only positive integer solution to the equation $24 = 6x + 9y$ is $(x, y) = (1, 2)$. In such a covering there would be one vertex left out of one of the copies of $\overline{C}_6$. It would be impossible to use all edges at this vertex with the remaining copies of $C_6$ and $\overline{C}_6$. Thus, the best possible cardinality of a padding is 6. Let $V(K_7) = \{1, ..., 7\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_7$, with padding of $\{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\}$.

Example 11. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_8$ has a padding of cardinality 2.

Let $V(K_8) = \{1, ..., 8\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_8$, with padding of $\{\{1, 8\}, \{3, 5\}\}$.

Example 12. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_9$ has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of $K_9$ with a padding of cardinality 3 can be obtained by starting with a $C_6$-decomposition of $K_9$ which exists by Theorem 1.5. One of the copies of $C_6$ contained in this decomposition can be transformed into a copy of $\overline{C}_6$ by carefully adding 3 edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of $K_9$ where the three added edges form the padding.

Example 13. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ has a padding of cardinality 3.

A $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ with a padding of cardinality 3 can be obtained by starting with a $\overline{C}_6$-decomposition of $K_{10}$. One copy of $\overline{C}_6$ can be transformed into two copies of $C_6$ by carefully adding three edges. This gives us the desired $(C_6, \overline{C}_6)$-multicovering of $K_{10}$ where the three added edges form the padding.

Example 14. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{11}$ has a padding of cardinality 2.

Let $V(K_{11}) = \{1, ..., 11\}$. The following is a minimum $(C_6, \overline{C}_6)$-multicovering of $K_{11}$, with padding of $\{\{3, 4\}, \{8, 11\}\}$.

Example 15. A minimum $(C_6, \overline{C}_6)$-multicovering of $K_{17}$ has a padding of cardinality 2.

Let $V(K_{17}) = \{1, ..., 17\}$. Apply Theorem 1.5 and let $B_1$ be a $C_6$-decomposition on the copy of $K_9$ formed by the subgraph induced by the vertices $\{9, 10, 11, 17\}$. Apply Theorem 1.4 and let $B_2$ be a $C_6$-decomposition of the copy of $K_{11, 8}$ formed by the subgraph of $K_{17}$ with vertex bipartition.
Lemma 4.1. If \( n \equiv 2 \pmod{6} \) with \( n \geq 8 \), then \( K_n \) admits a minimum \((C_6, \overline{C}_6)\)-multicovering with a padding of cardinality 2.

Proof. Let \( n = 6x + 2 \) for some integer \( x \geq 1 \). Notice that \( K_{6x+2} \cong K_8 \lor \left( \lor_{i=1}^{x-1} K_6 \right) \). Each of the \( x-1 \) copies of \( K_6 \) admit a \((C_6, \overline{C}_6)\)-multidecomposition by Lemma 2.5. The copy of \( K_8 \) admits a \((C_6, \overline{C}_6)\)-multicovering with a padding of cardinality 2 by Example 11. The remaining edges form edge disjoint copies of \( K_{6,6} \) or \( K_{6,8} \), each of which admit a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multicovering of \( K_n \).

Lemma 4.2. If \( n \equiv 5 \pmod{6} \) with \( n \geq 11 \), then \( K_n \) admits a minimum \((C_6, \overline{C}_6)\)-multicovering with a padding of cardinality 2.

Proof. Let \( n = 6x + 5 \) for some integer \( x \geq 1 \). The proof breaks into two cases.

Case 1: \( x = 2k \) for some integer \( k \geq 1 \). Notice that \( K_{12k+5} \cong K_1 \lor K_4 \lor \left( \lor_{i=1}^{k} K_{12} \right) \). One copy of \( K_{17} \) is formed by \( K_1 \lor K_4 \lor K_{12}, \) and admits a \((C_6, \overline{C}_6)\)-multicovering with a padding of cardinality 2 by Example 15. The \( k-1 \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admit a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge disjoint copies of \( K_{12,12} \) or \( K_{4,12}, \) each of which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multicovering of \( K_n \).

Case 2: \( x = 2k + 1 \) for some integer \( k \geq 1 \). Notice that \( K_{12k+11} \cong K_1 \lor K_4 \lor K_6 \lor \left( \lor_{i=1}^{k} K_{12} \right). \) One copy of \( K_{11} \) is formed by \( K_1 \lor K_4 \lor K_6, \) and admits a \((C_6, \overline{C}_6)\)-multicovering with a padding of cardinality 2 by Example 14. The \( k \) copies of \( K_{13} \) formed by \( K_1 \lor K_{12} \) admit a \((C_6, \overline{C}_6)\)-multidecomposition by Example 1. The remaining edges form edge disjoint copies of \( K_{12,12}, K_{4,12}, \) or \( K_{6,12}, \) each of which admits a \( C_6 \)-decomposition by Theorem 1.4. Thus, we obtain the desired \((C_6, \overline{C}_6)\)-multicovering of \( K_n \).

Combining Lemmas 4.1 and 4.2, we have proven Theorem 1.3.

5. Conclusion

The cardinalities of the leaves of maximum \((C_6, \overline{C}_6)\)-multipackings and paddings of minimum \((C_6, \overline{C}_6)\)-multicoverings of \( K_n \) have been characterized. However, the achievable structures of these leaves and paddings are still yet to be characterized. This leads to the following open question.

Open Problem 1. For each positive integer \( n, \) characterize all possible graphs \((\text{multigraphs})\) which are leaves \((\text{paddings})\) of a \((C_6, \overline{C}_6)\)-multipacking \((\text{multicovering})\) of \( K_n \).
Furthermore, it would be of interest to know when a $(C_6, \overline{C_6})$-multidecomposition of $K_n$ exists with $p$ copies of $C_6$ and $q$ copies of $\overline{C_6}$ where $(p, q)$ is any solution to the equation $6p + 9q = \binom{n}{2}$. This leads to the following open problem.

**Open Problem 2.** Let $p, q$ and $n$ be positive integers for which $6p + 9q = \binom{n}{2}$. Determine whether a $(C_6, \overline{C_6})$-multidecomposition of $K_n$ exists with $p$ copies of $C_6$ and $q$ copies of $\overline{C_6}$.

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**References**


